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Properties of the Equilibrium Revenues in Buy Price Auctions

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Abstract

There are many equilibria in second-price sealed-bid auctions with a buy price. This paper analyzes multiple equilibria and then characterizes the set of equilibrium revenues. A risk-neutral seller who faces risk-neutral bidders can always obtain a positive equilibrium revenue by introducing a buy price properly. Our result gives a seller a new reason to use a buy price in Internet auctions.

JEL classification: D44

Key words: Auction, buy price, multiple equilibria

1 Introduction

Most research on buy price auctions explains the effects of a buy price from the viewpoint of seller's risk attitude or bidders' risk attitudes. Budish and Takeyama (2001) consider second-price sealed-bid auctions with a buy price and show that a risk-neutral seller can improve her expected revenue by properly introducing a buy price when bidders are risk-averse. Hidvégi, Wang and Whinston (2006) and Reynolds and Wooders (2009) examine English auctions with a buy price. Though Budish and Takeyama (2001) use a two-bidder two-type framework, they consider a more general framework, where there are n bidders whose types are continuously distributed, and prove that seller's risk-aversion and bidders' risk-aversion play key roles in improving a seller's expected revenue.

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A Buy-It-Now price option has a similar feature to that of a buy price option.¹ In Buy-It-Now price auctions, seller's risk aversion or bidders' risk aversion also accounts for the effects on a seller's expected revenue. Mathews and Katzman (2006) consider second-price sealed-bid auctions with a Buy-It-Now price and show that a risk-averse seller who faces risk-neutral bidders can improve her expected revenue by introducing a Buy-It-Now price. Reynolds and Wooders (2009) analyze English auctions with a Buy-It-Now price and give similar results to those of buy price auctions.

When both a seller and bidders are risk-neutral, however, a seller cannot increase her expected revenue by introducing a buy price or a Buy-It-Now price and obtains at most the same expected revenue as those of second-price sealed-bid auctions or English auctions. In this sense, it is no substantial benefit for a seller to use a buy price or a Buy-It-Now price when she conducts second-price sealed-bid auctions or English auctions.²

We consider second-price sealed-bid auctions with a buy price. There are two bidders whose types are uniformly distributed. In a private value environment, we reveal an advantage of a buy price when both a seller and bidders are risk-neutral. For this purpose, we attempt to examine the effects of a buy price from another perspective. Inami (2011) highlights that there is an asymmetric equilibrium in addition to a symmetric equilibrium in second-price sealed-bid auctions with a buy price. We focus on multiple equilibria in the analysis. To examine an equilibrium as much as possible, for example, we consider an equilibrium composed of a strategy such that a bidder whose all types belong to one interval bid the same bid.

Milgrom (1981) points out that there are many equilibria in second-price sealed-bid auctions without a buy price.³ We pay attention to this fact and contrast between the cases of second-price sealed-bid auctions with and without a buy price. In second-price sealed-bid auctions without a buy price, we show that the set of equilibrium revenues is a closed interval.⁴ To obtain this result, we construct an equilibrium whose expected revenue corresponds to each element in that interval. A truth-telling strategy equilibrium supports the maximum of equilibrium revenues. And the minimum of equilibrium revenues is zero.⁵

Similarly, we characterize the set of equilibrium revenues for each buy price auction. In the same way as second-price sealed-bid auctions without a buy price, we show that each set of equilibrium revenues is also a closed interval. A symmetric equilibrium supports the maximum of equilibrium revenues for each buy price auction. We show that a seller can

¹In a buy price auction, bidders can submit a buy price throughout the auction. On the other hand, in a Buy-It-Now price auction, bidders can submit a Buy-It-Now price only before the auction starts.

²When risk-neutral bidders need to pay a participation cost and sequentially arrive at an auction, there is a case in which a risk-neutral seller can increase her expected revenue by introducing a Buy-It-Now price. See Che (2011).

³Milgrom (1981) considers the case in which there are two bidders whose types are continuously distributed. Maskin and Riley (1985) show that there are many equilibria even in a two-bidder two-type framework.

⁴In the case of more than two bidders, the set of equilibrium revenues is also a closed interval. See Inami (2018) for details.

⁵It is well-known that there exists an equilibrium in which a seller's expected revenue is zero in second-price sealed-bid auctions without a buy price.

always obtain a positive equilibrium revenue by introducing a buy price properly.

According to our knowledge, a limited number of studies explain the effects of a buy price or a Buy-It-Now price without considering a seller's risk attitude and bidders' risk attitudes. Mathews (2004) brings a possibility that a seller or bidders discount a future revenue or future payoffs into the analysis. Mathews (2004) examines second-price sealed-bid auctions with a Buy-It-Now price and shows that an impatient seller can increase her expected revenue by introducing a Buy-It-Now price. Shunda (2009) also considers second-price sealed-bid auctions with a Buy-It-Now price. Shunda (2009) examines the case in which bidders have reference-dependent preferences and shows that a seller chooses a Buy-It-Now price that a bidder exercises with positive probability.

There is a bit of research focusing on multiple equilibria.⁶ Blume and Heidhues (2004) consider second-price sealed-bid auctions where there are at least three bidders. They characterize the set of all equilibria and show that a seller can make an equilibrium unique by properly introducing a reserve price. Plum (1992) characterizes all equilibria in two-bidder auctions with several payment rules, in which the winner pays a convex combination of the highest bid and the second-highest bid. Plum (1992) shows that any equilibrium is composed of continuously differentiable and strictly monotone increasing strategies in all but second-price sealed-bid auctions.

The rest of this paper is organized as follows. Section 2 describes the model. In Section 3, we derive the set of equilibrium revenues for each second-price sealed-bid auction with a buy price. Section 4 reveals the effects of a buy price on equilibrium revenues. And Section 5 concludes.

2 Preliminaries

2.1 The model

A risk-neutral seller puts one item up for auction. Two risk-neutral bidders participate in the auction. Each bidder's type is independently drawn from a uniform distribution. The bidders evaluate the item, depending only on their type.

The rule of second-price sealed-bid auctions with a buy price requires an explanation. A seller sets a buy price before an auction starts. Bidders observe the buy price and then submit a bid. If no one bids the buy price, then the highest bidder wins the auction and pays the second-highest bid to the seller. If only one bidder bids the buy price, then he certainly wins the auction but must pay it to the seller. If bidders bid the same amount (it may be the buy

⁶In a common value environment, Bikhchandani and Riley (1991) consider not only a symmetric equilibrium but also asymmetric equilibria. They compare the seller's expected revenues in second-price sealed-bid auctions with those of English auctions. Lizzeri and Persico (2000) examine various kinds of auction formats except for second-price sealed-bid auctions in an interdependent value environment. They derive necessary assumptions to obtain a uniqueness result.

price), the winner, who is determined with an equal probability, pays the other bidder's bid to the seller.

We formalize second-price sealed-bid auctions with a buy price as follows. A seller sets a buy price $B \in [0, +\infty)$. Let $N = \{1, 2\}$ be the set of bidders. For each bidder i , let $T_i = [0, \omega]$ be the set of types. For each bidder i , let $A_i = [0, \bar{b}] \cup \{B\}$ be the set of actions.⁷ The bid \bar{b} is the highest possible bid except the buy price B . In Internet auctions, bidders are not allowed to bid above a buy price B . Then, it is assumed that the bid \bar{b} is less than the buy price B .⁸ For each bidder i , a payoff function is $u_i : A \times T_i \rightarrow \mathbb{R}$, where $A = A_i \times A_j$ ($j \neq i$). For each bidder i , a payoff is determined by

$$u_i(a; t_i) = \begin{cases} t_i - a_j & \text{if } a_i \neq B \text{ and } a_i > a_j \\ t_i - B & \text{if } a_i = B \text{ and } a_i > a_j \\ \frac{t_i - a_j}{2} & \text{if } a_i = a_j \text{ and} \\ 0 & \text{if } a_i < a_j, \end{cases}$$

given $t_i \in T_i$ and $a \in A$.

For each bidder i , a strategy is $\sigma_i : T_i \rightarrow A_i$. We consider pure and non-decreasing strategies. We adopt Bayesian Nash equilibrium as a solution concept. The strategy profile $\sigma = (\sigma_i(\cdot), \sigma_j(\cdot))$ is a Bayesian Nash equilibrium if for all bidder i , all $t_i \in T_i$, and all $a'_i \in A_i$,

$$E[u_i(a; t_i) | \sigma_i(\cdot), \sigma_j(\cdot), f(\cdot)] \geq E[u_i(a'_i, a_j; t_i) | \sigma_j(\cdot), f(\cdot)],$$

where $f(\cdot)$ is a common prior belief about the other bidder's type.

There are many kinds of equilibria in second-price sealed-bid auctions without a buy price B .⁹ We then restrict attention to a truth-telling strategy equilibrium and equilibria with properties: there exists one interval of types such that one bidder submits the infimum of the interval for all types in that interval and the other bidder submits the supremum of the interval for all types in that interval; both bidders submit their own types for the rest of types.¹⁰ In the analysis, for example, we take account of the following equilibrium. Fix $0 \leq \underline{t} < \bar{t} \leq \omega$ arbitrarily. Then, consider the equilibrium $\sigma' = (\sigma'_i(\cdot), \sigma'_j(\cdot))$ such that

$$\sigma'_i(t_i) = \begin{cases} t_i & \text{if } 0 \leq t_i \leq \underline{t} \\ \underline{t} & \text{if } \underline{t} < t_i < \bar{t} \\ t_i & \text{if } \bar{t} \leq t_i \leq \omega \end{cases}$$

⁷Of course, we can adopt other set of actions, for example $[0, B]$. In this case, however, we face non-existence of an equilibrium under some buy price B .

⁸Moreover, we consider the case in which the bid \bar{b} is arbitrarily close to the buy price B .

⁹Blume and Heidhues (2001) characterize the set of equilibria in the case of two bidders. However, we could not find their article at: <http://www.wiwi.uni-bonn.de/heidhues/Paper/characterization2.PDF>, accessed on May 27, 2021.

¹⁰Blume and Heidhues (2004) point out the existence of such equilibria in the case of two bidders.

and

$$\sigma'_j(t_j) = \begin{cases} t_j & \text{if } 0 \leq t_j \leq \underline{t} \\ \bar{t} & \text{if } \underline{t} < t_j < \bar{t} \\ t_j & \text{if } \bar{t} \leq t_j \leq \omega. \end{cases}$$

In the equilibrium σ' , there exists the interval (\underline{t}, \bar{t}) of types, which satisfies the above-mentioned properties.

2.2 The set of equilibrium revenues in second-price auctions without a buy price B

Our main purpose is to give a seller a new reason to use a buy price B in Internet auctions. For this purpose, we need to reveal what kinds of advantages a buy price B has. We contrast between the cases of second-price sealed-bid auctions with and without a buy price B . Specifically speaking, we derive the set of equilibrium revenues for each case and then make a detailed comparison.

At first we consider second-price sealed-bid auctions without a buy price B . We then have the following proposition.

Proposition 1. *Consider second-price sealed-bid auctions without a buy price B . Then, the set of equilibrium revenues is $[0, \frac{\omega}{3}]$.*

Proof. See Appendix.

To obtain the set of equilibrium revenues, we have constructed an equilibrium in practice. The set of equilibrium revenues is a closed interval. The minimum equilibrium revenue is zero. And the maximum equilibrium revenue is supported by a truth-telling strategy equilibrium.

As a seller cannot know which equilibrium emerges, she risks the possibility that an equilibrium revenue is zero in second-price sealed-bid auctions without a buy price B .

3 The set of equilibrium revenues in second-price auctions with a buy price B

In this section, we consider second-price sealed-bid auctions with a buy price B . Especially, we focus on second-price sealed-bid auctions with a buy price $B \in (0, \omega]$. (In Appendix, we analyze the rest of buy price auctions—a second-price sealed-bid auction with the buy price $B = 0$ and second-price sealed-bid auctions with a buy price $B \in (\omega, +\infty)$, respectively.) We pick up one strategy profile and then derive the set of equilibrium revenues by modifying it.¹¹

¹¹For a second-price sealed-bid auction with the buy price $B = 0$, the set of equilibrium revenues is $\{0\}$. For second-price sealed-bid auctions with a buy price $B \in (\omega, +\infty)$, the sets of equilibrium revenues are equivalent to those of second-price sealed-bid auctions without a buy price B . See Appendix for details.

Once a seller sets a buy price B , a bidder whose type is greater than the buy price B cannot submit his own type. In second-price sealed-bid auctions with a buy price B , thus, we pay attention to strategy profiles such that: a bidder whose type is not less than the buy price B submits the bid \bar{b} or the buy price B ; a bidder whose type is less than the buy price B submits a bid in a similar way to those of second-price sealed-bid auctions without a buy price B .¹²

One of the most basic strategy profiles that would be expected to be an equilibrium is a strategy profile that both bidders' all types which are not less than the buy price B submit it. Surprisingly, the strategy profile is not an equilibrium.

Proposition 2. *Consider second-price sealed-bid auctions with a buy price $B \in (0, \omega]$. Then, any strategy profile such that both bidders' all types which are not less than the buy price B submit it is not a Bayesian Nash equilibrium.*

Proof. See Appendix.

To derive the set of equilibrium revenues, by Proposition 2, we need to find other strategy profile, where bidders' types which are not less than the buy price B do not always submit it.

3.1 The case of second-price auctions with a buy price $B \in (0, \frac{\omega}{2}]$

When we derive the set of equilibrium revenues in second-price sealed-bid auctions with a buy price $B \in (0, \omega]$, we mainly look at two strategy profiles. For the analysis, we divide second-price sealed-bid auctions with a buy price $B \in (0, \omega]$ into two cases. One is second-price sealed-bid auctions with a buy price $B \in (0, \frac{\omega}{2}]$. The other is second-price sealed-bid auctions with a buy price $B \in (\frac{\omega}{2}, \omega]$.

First we consider second-price sealed-bid auctions with a buy price $B \in (0, \frac{\omega}{2}]$. Fix $b \in [0, \bar{b}]$ arbitrarily. Then, consider the strategy profile $\sigma^\# = (\sigma_i^\#(\cdot), \sigma_j^\#(\cdot))$ such that

$$\sigma_i^\#(t_i) = \begin{cases} 0 & \text{if } 0 \leq t_i < b \\ t_i & \text{if } b \leq t_i \leq \bar{b} \\ \bar{b} & \text{if } B \leq t_i \leq t_i^\# \\ B & \text{if } t_i^\# < t_i \leq \omega \end{cases}$$

¹²Specifically speaking, both bidders submit their types for all types or there exists one interval of types such that one bidder submits the infimum of the interval for all types in the interval and the other bidder submits the supremum of the interval for all types in the interval; both bidders submit their types for the rest of types.

and

$$\sigma_j^\sharp(t_j) = \begin{cases} b & \text{if } 0 \leq t_j < b \\ t_j & \text{if } b \leq t_j \leq \bar{b} \\ \bar{b} & \text{if } B \leq t_j \leq t_j^\sharp \\ B & \text{if } t_j^\sharp < t_j \leq \omega. \end{cases}$$

Here let $t_i^\sharp := \frac{\omega B - b^2}{\omega - B}$ and $t_j^\sharp := \frac{\omega B + b^2}{\omega - B}$, respectively. Once we fix the type $b \in [0, \bar{b}]$, then the corresponding strategy profile σ^\sharp is uniquely determined. When bidders play the strategies $\sigma_i^\sharp(\cdot)$ and $\sigma_j^\sharp(\cdot)$, bidders' types which are not less than the buy price B but not greater than the thresholds do not submit the buy price B .

The strategy profile σ^\sharp is an equilibrium, which is shown in Appendix. And the seller obtains the equilibrium revenue, which is given by

$$\frac{1}{6\omega^2} \{-2b^3 + 3(t_i^\sharp - t_j^\sharp)b^2 + 2B^3 - 3(t_i^\sharp + t_j^\sharp)B^2 + 6\omega^2 B\}.$$

Here let $E[R^\sharp(b)] := \frac{1}{6\omega^2} \{-2b^3 + 3(t_i^\sharp - t_j^\sharp)b^2 + 2B^3 - 3(t_i^\sharp + t_j^\sharp)B^2 + 6\omega^2 B\}$. The equilibrium revenue $E[R^\sharp(\cdot)]$ is a function of the type b on the domain $[0, \bar{b}]$. In a similar way to those of second-price sealed-bid auctions without a buy price B , we have the following proposition.

Proposition 3. *Consider second-price sealed-bid auctions with a buy price $B \in (0, \frac{\omega}{2}]$. Then, the set of equilibrium revenues is*

$$\begin{aligned} [E[R^\sharp(\bar{b})], E[R^\sharp(0)]] & \quad \text{if } B \in (0, (\sqrt{2} - 1)\omega] \quad \text{and} \\ [E[R^\sharp(\hat{b})], E[R^\sharp(0)]] & \quad \text{if } B \in ((\sqrt{2} - 1)\omega, \frac{\omega}{2}]. \end{aligned}$$

Here let $\hat{b} := \sqrt{\omega^2 - 2\omega B}$.

Proof. See Appendix.

The set of equilibrium revenues is a closed interval. In second-price sealed-bid auctions with a buy price $B \in (0, (\sqrt{2} - 1)\omega]$, both $t_i^\sharp \in [B, \omega]$ and $t_j^\sharp \in [B, \omega]$ are well-defined. In second-price sealed-bid auctions with a buy price $B \in ((\sqrt{2} - 1)\omega, \frac{\omega}{2}]$, however, there is a case in which $t_j^\sharp \in [B, \omega]$ is not well-defined. Thus, we need to modify the domain of the equilibrium revenue $E[R^\sharp(\cdot)]$.

3.2 The case of second-price auctions with a buy price $B \in (\frac{\omega}{2}, \omega]$

Now, we consider second-price sealed-bid auctions with a buy price $B \in (\frac{\omega}{2}, \omega]$. In these auctions, the strategy profile σ^\sharp is not an equilibrium. Instead, we pay attention to other strategy profile. Fix $b \in [0, \bar{b}]$ arbitrarily. Then, consider the strategy profile $\sigma^\natural = (\sigma_i^\natural(\cdot), \sigma_j^\natural(\cdot))$

such that

$$\sigma_i^{\sharp}(t_i) = \begin{cases} 0 & \text{if } 0 \leq t_i < b \\ t_i & \text{if } b \leq t_i \leq \bar{b} \\ \bar{b} & \text{if } B \leq t_i \leq \omega \end{cases}$$

and

$$\sigma_j^{\sharp}(t_j) = \begin{cases} b & \text{if } 0 \leq t_j < b \\ t_j & \text{if } b \leq t_j \leq \bar{b} \\ \bar{b} & \text{if } B \leq t_j \leq \omega. \end{cases}$$

Once we fix the type $b \in [0, \bar{b}]$, then the corresponding strategy profile σ^{\sharp} is uniquely determined. When bidders play the strategies $\sigma_i^{\sharp}(\cdot)$ and $\sigma_j^{\sharp}(\cdot)$, bidders' types which are not less than the buy price B do not submit it at all.

The strategy profile σ^{\sharp} is an equilibrium. And the seller obtains the equilibrium revenue, which is given by

$$\frac{1}{3\omega^2}(-b^3 + B^3 - 3\omega B^2 + 3\omega^2 B).$$

Here let $E[R^{\sharp}(b)] := \frac{1}{3\omega^2}(-b^3 + B^3 - 3\omega B^2 + 3\omega^2 B)$. As well as the case of the equilibrium σ^{\sharp} , the equilibrium revenue $E[R^{\sharp}(\cdot)]$ is a function of the type b on the domain $[0, \bar{b}]$. We then have the following proposition.

Proposition 4. *Consider second-price sealed-bid auctions with a buy price $B \in (\frac{\omega}{2}, \omega]$. Then, the set of equilibrium revenues is $[E[R^{\sharp}(\tilde{b})], E[R^{\sharp}(0)]]$ if $B \in (\frac{\omega}{2}, \omega]$. Here let $\tilde{b} := \sqrt{-\omega^2 + 2\omega B}$.*

Proof. See Appendix.

The set of equilibrium revenues is a closed interval. In second-price sealed-bid auctions with a buy price $B \in (\frac{\omega}{2}, \omega]$, we need to restrict the domain of the equilibrium revenue $E[R^{\sharp}(\cdot)]$, which guarantees the existence of an equilibrium.

4 The effects of a buy price B on equilibrium revenues

We have derived for each buy price auction the corresponding set of equilibrium revenues. In this section, we show that introducing a buy price gives a seller a distinct advantage. For this purpose, we focus on the minimum of equilibrium revenues.

At first, we focus on the maximum of equilibrium revenues of each second-price sealed-bid auction with a buy price B . Here let $ER_{\max}^B(B)$ be a function of the maximum equilibrium revenues in second-price sealed-bid auctions with a buy price B . By Propositions 3 and 4 and

the results in Appendix, we have

$$ER_{\max}^B(B) := \begin{cases} E[R^\sharp(0)] & \text{if } 0 \leq B \leq \frac{\omega}{2} \\ E[R^\natural(0)] & \text{if } \frac{\omega}{2} < B \leq \omega \\ \frac{\omega}{3} & \text{if } \omega < B. \end{cases}$$

$ER_{\max}^B(\cdot)$ is continuous at the buy price $B = \frac{\omega}{2}$ because $E[R^\sharp(0)] = \frac{7\omega}{24} = E[R^\natural(0)]$.

To illustrate, we provide an example.

Example 1. Suppose that bidders' types are distributed on the interval $[0, 1]$. Then, the maximum of equilibrium revenues is given by

$$ER_{\max}^B(B) = \begin{cases} \frac{-B^4 - 2B^3 - 3B^2 + 3B}{3(1-B)} & \text{if } 0 \leq B \leq \frac{1}{2} \\ \frac{B^3 - 3B^2 + 3B}{3} & \text{if } \frac{1}{2} < B \leq 1 \\ \frac{1}{3} & \text{if } 1 < B. \end{cases}$$

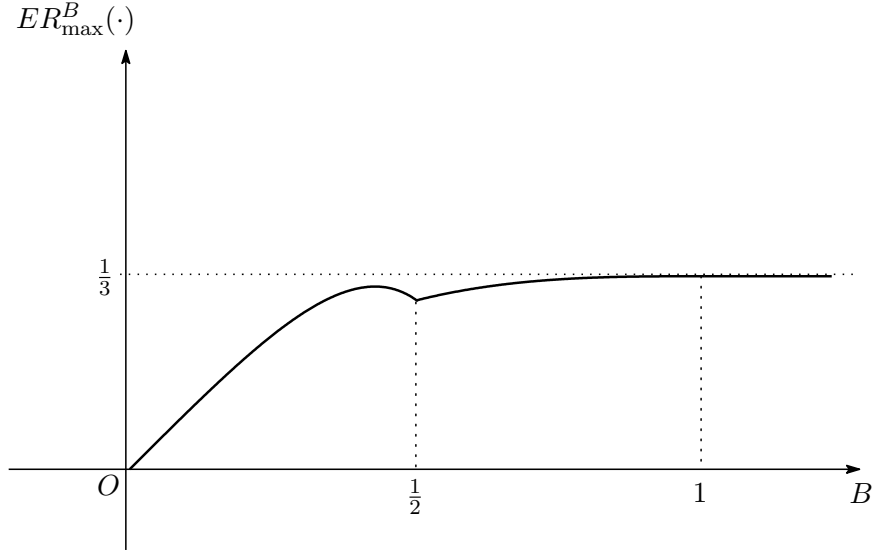


Figure 1: The maximum of equilibrium revenues $ER_{\max}^B(B)$

In Figure 1, $ER_{\max}^B(\cdot)$ is not monotone increasing with respect to a buy price $B \in [0, \frac{1}{2}]$. On the other hand, $ER_{\max}^B(\cdot)$ is monotone increasing with respect to a buy price $B \in (\frac{1}{2}, \omega]$. Moreover, a seller cannot obtain a higher equilibrium revenue at an equilibrium where bidders submit a buy price B than at a truth-telling strategy equilibrium in second-price sealed-bid auctions without a buy price B .

Next, we focus on the minimum of equilibrium revenues of each second-price sealed-bid auction with a buy price B . Here let $ER_{\min}^B(B)$ be a function of the minimum equilibrium revenues in second-price sealed-bid auctions with a buy price B . By Propositions 3 and 4 and

the results in Appendix, we have

$$ER_{\min}^B(B) := \begin{cases} E[R^\sharp(\bar{b})] & \text{if } 0 \leq B \leq (\sqrt{2} - 1)\omega \\ E[R^\sharp(\hat{b})] & \text{if } (\sqrt{2} - 1)\omega < B \leq \frac{\omega}{2} \\ E[R^\sharp(\tilde{b})] & \text{if } \frac{\omega}{2} < B \leq \omega \\ 0 & \text{if } \omega < B. \end{cases}$$

Note that $\hat{b} = \sqrt{\omega^2 - 2\omega B}$ and $\tilde{b} = \sqrt{-\omega^2 + 2\omega B}$, respectively. $ER_{\min}^B(\cdot)$ is continuous at the buy price $B = (\sqrt{2} - 1)\omega$ because $\bar{b} = (\sqrt{2} - 1)\omega = \hat{b}$. And $ER_{\min}^B(\cdot)$ is continuous at the buy price $B = \frac{\omega}{2}$ because $E[R^\sharp(0)] = \frac{7\omega}{24} = E[R^\sharp(\omega)]$.

Now we are ready to show the main theorem.

Theorem 1. *Consider second-price sealed-bid auctions with a buy price $B \in [0, +\infty)$. Then, a seller can always obtain a positive equilibrium revenue by introducing a buy price B properly.*

Proof. See Appendix.

In the case that a seller sets the buy price $B = 0$, she obtains zero equilibrium revenue. If the seller sets a buy price B that is greater than the highest possible type ω , no bidder intends to submit the buy price B at any equilibrium. Thus, a rational seller does not set such buy prices B .

We give an illustrative example.

Example 2. Suppose that bidders' types are distributed on the interval $[0, 1]$. Then, the minimum of equilibrium revenues is given by

$$ER_{\min}^B(B) = \begin{cases} \frac{-B^4 - B^3 - B^2 + B}{1 - B} & \text{if } 0 \leq B \leq \sqrt{2} - 1 \\ \frac{-3(\sqrt{1-2B})^4 - (\sqrt{1-2B})^3(1-B) - B^4 - 2B^3 - 3B^2 + 3B}{3(1-B)} & \text{if } \sqrt{2} - 1 < B \leq \frac{1}{2} \\ \frac{-(\sqrt{-1+2B})^3 + B^3 - 3B^2 + 3B}{3} & \text{if } \frac{1}{2} < B \leq 1 \\ 0 & \text{if } 1 < B. \end{cases}$$

In Figure 2, $ER_{\min}^B(\cdot)$ is not monotone increasing with respect to a buy price $B \in [0, \frac{1}{2}]$. On the other hand, $ER_{\min}^B(\cdot)$ is monotone decreasing with respect to a buy price $B \in (\frac{1}{2}, \omega]$. In Figure 2, $ER_{\min}^B(\cdot)$ is absolutely positive for any buy price $B \in (0, 1)$.

5 Conclusion

We have investigated how the introduction of a buy price affects a seller's equilibrium revenues. In the analysis, we have looked at not only a truth-telling strategy equilibrium but also elaborate pure strategy equilibria. For each buy price auction, we have characterized the set of equilibrium revenues. We have shown that a seller can always obtain a positive equilibrium revenue by setting a buy price properly. On the other hand, the minimum of equilibrium

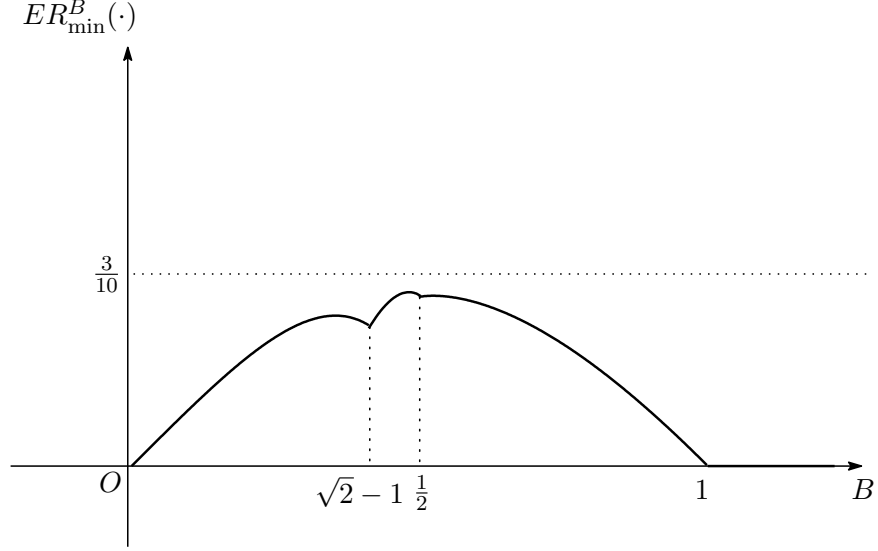


Figure 2: The minimum of equilibrium revenues $ER_{\min}^B(B)$

revenues is zero in second-price sealed-bid auctions without a buy price. By introducing a buy price, thus, a seller can avoid a risk that she may obtain zero equilibrium revenue. Our results provide a seller a new reason to use a buy price in Internet auctions.

Appendix

Proof of Proposition 1

We organize the proof in two steps.

Step 1. We show that each element in the interval $[0, \frac{\omega}{3}]$ is an equilibrium revenue.

Fix $b \in [0, \omega]$ arbitrarily. Then, consider the strategy profile $\sigma^\dagger = (\sigma_i^\dagger(\cdot), \sigma_j^\dagger(\cdot))$ such that

$$\sigma_i^\dagger(t_i) = \begin{cases} 0 & \text{if } 0 \leq t_i \leq b \\ t_i & \text{if } b < t_i \leq \omega \end{cases}$$

and

$$\sigma_j^\dagger(t_j) = \begin{cases} b & \text{if } 0 \leq t_j \leq b \\ t_j & \text{if } b < t_j \leq \omega. \end{cases}$$

We show that the strategy profile σ^\dagger is an equilibrium.¹³ Consider incentive constraints of

¹³In Blume and Heidhues (2001), it is shown that a strategy profile like the strategy profile σ^\dagger is an equilibrium.

bidder i . For all $t_i \in [0, b]$, he obtains at most zero expected payoff by submitting other bids, for example the bid b . For all $t_i \in (b, \omega]$, he only reduces his expected payoff if he submits a bid that is greater or less than his type t_i . Next, consider incentive constraints of bidder j . For all $t_j \in [0, b]$, he only reduces his expected payoff if he submits a bid that is greater or less than the bid b . For all $t_j \in (b, \omega]$, he cannot improve his expected payoff by submitting a bid that is greater or less than his type t_j . Thus, the strategy profile σ^\dagger is an equilibrium.

At the equilibrium σ^\dagger , the seller receives the equilibrium revenue

$$\begin{aligned} & \int_b^\omega \left(\int_{t_i}^\omega t_i f(t_j) dt_j \right) f(t_i) dt_i + \int_0^b \left(\int_b^\omega b f(t_i) dt_i \right) f(t_j) dt_j + \int_b^\omega \left(\int_{t_j}^\omega t_j f(t_i) dt_i \right) f(t_j) dt_j \\ &= \frac{\omega^3 - b^3}{3\omega^2}. \end{aligned}$$

By the way of construction of the strategy profile σ^\dagger , each element in the interval $[0, \frac{\omega}{3}]$ is an equilibrium revenue.

Step 2. We show that the equilibrium revenue $\frac{\omega}{3}$ is the maximum equilibrium revenue.

We consider whether there exists an equilibrium that yields a greater equilibrium revenue than that of a truth-telling strategy equilibrium.

At any equilibrium, there are at most one interval (\underline{t}, \bar{t}) of types in which one bidder's all types in the interval (\underline{t}, \bar{t}) submit the bid \underline{t} , and the other bidder's all types in the interval (\underline{t}, \bar{t}) submit the bid \bar{t} . Fix $0 \leq \underline{t} \leq \bar{t} \leq \omega$ arbitrarily. Then, consider the strategy profile $\sigma' = (\sigma'_i(\cdot), \sigma'_j(\cdot))$ such that

$$\sigma'_i(t_i) = \begin{cases} t_i & \text{if } 0 \leq t_i \leq \underline{t} \\ \underline{t} & \text{if } \underline{t} < t_i < \bar{t} \\ t_i & \text{if } \bar{t} \leq t_i \leq \omega \end{cases}$$

and

$$\sigma'_j(t_j) = \begin{cases} t_j & \text{if } 0 \leq t_j \leq \underline{t} \\ \bar{t} & \text{if } \underline{t} < t_j < \bar{t} \\ t_j & \text{if } \bar{t} \leq t_j \leq \omega. \end{cases}$$

Consider incentive constraints of bidder i . For all $t_i \in [0, \underline{t}]$, he reduces his expected payoff by submitting a bid that is greater or less than his type t_i . For all $t_i \in (\underline{t}, \bar{t})$, he obtains at most the same expected payoff even by submitting other bids. For all $t_i \in [\bar{t}, \omega]$, he reduces his expected payoff by submitting a bid that is greater or less than his type t_i . Next, consider incentive constraints of bidder j . For all $t_j \in [0, \underline{t}]$, he reduces his expected payoff by submitting a bid that is greater or less than his type t_j . For all $t_j \in (\underline{t}, \bar{t})$, he obtains at most the same expected payoff by submitting other bids. For all $t_j \in [\bar{t}, \omega]$, he reduces his

expected payoff by submitting a bid that is greater or less than his type t_j . From the above argument, the strategy profile σ' is an equilibrium.

Here let $E[R^*]$ and $E[R']$ be the equilibrium revenue of a truth-telling strategy equilibrium and that of the equilibrium σ' , respectively. We can find clear differences in a seller's expected revenue for three cases: (i) Both bidders' types are in the interval (\underline{t}, \bar{t}) . (ii) Bidder i 's type is in the interval (\underline{t}, \bar{t}) and bidder j 's type is not less than the type \bar{t} . (iii) Bidder i 's type is not less than the type \bar{t} and bidder j 's type is in the interval (\underline{t}, \bar{t}) . Note that a winner of the cases (ii) and (iii) does not change, respectively. Then, we have

$$\begin{aligned} E[R^*] - E[R'] &= \int_{\underline{t}}^{\bar{t}} \left(\int_{t_j}^{\omega} t_j f(t_i) dt_i \right) f(t_j) dt_j + \int_{\underline{t}}^{\bar{t}} \left(\int_{t_i}^{\omega} t_i f(t_j) dt_j \right) f(t_i) dt_i \\ &\quad - \int_{\underline{t}}^{\bar{t}} \left(\int_{\bar{t}}^{\omega} \bar{t} f(t_i) dt_i \right) f(t_j) dt_j - \int_{\underline{t}}^{\bar{t}} \left(\int_{\underline{t}}^{\omega} \underline{t} f(t_j) dt_j \right) f(t_i) dt_i \\ &= \frac{(\bar{t} - \underline{t})^3}{3\omega^2} \geq 0. \end{aligned}$$

Thus, $E[R^*]$ is greater than or equal to $E[R']$. In other words, $\frac{\omega}{3}$ is the maximum equilibrium revenue in second-price sealed-bid auctions without a buy price B .

A second-price auction with the buy price $B = 0$

We consider a second-price sealed-bid auction with the buy price $B = 0$. In this auction, a strategy profile in which both bidders' all types submit the buy price B is a unique equilibrium. Thus, the set of equilibrium revenues is $\{0\}$.

Second-price auctions with a buy price $B \in (\omega, +\infty)$

We consider second-price sealed-bid auctions with a buy price $B \in (\omega, +\infty)$. Since the buy price B is greater than the highest possible type ω , no one bids the buy price B at any equilibrium. Thus, the sets of equilibrium revenues are the same as those of second-price sealed-bid auctions without a buy price B .

Proof of Proposition 2

Suppose that both bidders' all types which are not less than the buy price B submit it. To be an equilibrium, one of the bidders whose type is less than the type \bar{b} must submit a bid that is not greater than own type. Without loss of generality, let bidder i whose type is less than the type \bar{b} submit a bid that is not greater than own type.

Consider incentive constraints of bidder i . We wonder whether for all $t_i \in [B, \omega]$,

$$\int_0^{\bar{b}} (t_i - B) f(t_j) dt_j + \frac{1}{2} \int_B^{\omega} (t_i - B) f(t_j) dt_j \geq \int_0^{\bar{b}} (t_i - t_j) f(t_j) dt_j \quad (1)$$

holds. The RHS of (1) is the most profitable deviation among all strategy profiles such that both bidders' all types which are not less than the buy price B submit it. Calculating (1), we have

$$\frac{1}{2\omega} \{(\omega - B)t_i + B^2 - \omega B - 2\bar{b}B + \bar{b}^2\} \geq 0.$$

Since it is assumed that the bid \bar{b} is arbitrarily close to the buy price B , we can evaluate \bar{b} at B . Substituting B for \bar{b} , we have

$$\frac{1}{2\omega} \{(\omega - B)t_i - \omega B\} \geq 0.$$

Here let

$$\phi(t_i) := (\omega - B)t_i - \omega B.$$

For $t_i \in [B, \omega]$, the function $\phi(\cdot)$ is increasing with respect to t_i . To be an equilibrium, thus, it is sufficient to show that (1) holds for $t_i = B$. However, we have

$$-B^2 \geq 0,$$

which does not hold.

From the above argument, any strategy profiles such that both bidders' all types which are not less than the buy price B submit it is not an equilibrium.

Proof of Proposition 3

We organize the proof in three steps.

Step 1. We show that a certain strategy profile is an equilibrium.

Fix $0 \leq \underline{t} \leq \bar{t} \leq \bar{b}$ arbitrarily. Then, consider the following strategy profile $\sigma^S = (\sigma_i^S(\cdot), \sigma_j^S(\cdot))$ such that

$$\sigma_i^S(t_i) = \begin{cases} t_i & \text{if } 0 \leq t_i \leq \underline{t} \\ \underline{t} & \text{if } \underline{t} < t_i < \bar{t} \\ t_i & \text{if } \bar{t} \leq t_i \leq \bar{b} \\ \bar{b} & \text{if } B \leq t_i \leq t_i^S \\ B & \text{if } t_i^S < t_i \leq \omega \end{cases}$$

and

$$\sigma_j^S(t_j) = \begin{cases} t_j & \text{if } 0 \leq t_j \leq \underline{t} \\ \bar{t} & \text{if } \underline{t} < t_j < \bar{t} \\ t_j & \text{if } \bar{t} \leq t_j \leq \bar{b} \\ \bar{b} & \text{if } B \leq t_j \leq t_j^S \\ B & \text{if } t_j^S < t_j \leq \omega. \end{cases}$$

Here let $t_i^S := \frac{\omega B - (\bar{t} - \underline{t})^2}{\omega - B}$ and $t_j^S := \frac{\omega B + (\bar{t} - \underline{t})^2}{\omega - B}$, respectively.

We show that the strategy profile σ^S is an equilibrium. Consider incentive constraints of bidder i . For all $t_i \in [0, \underline{t}]$, he reduces his expected payoff by submitting a bid that is greater or less than his type t_i . For all $t_i \in (\underline{t}, \bar{t})$, he obtains at most the same expected payoff even by submitting other bids. For all $t_i \in [\bar{t}, \bar{b}]$, he reduces his expected payoff by submitting a bid that is greater or less than his type t_i . For all $t_i \in [B, t_i^S]$,

$$\begin{aligned} & \int_0^{\underline{t}} (t_i - t_j) f(t_j) dt_j + \int_{\underline{t}}^{\bar{t}} (t_i - \bar{t}) f(t_j) dt_j + \int_{\bar{t}}^{\bar{b}} (t_i - t_j) f(t_j) dt_j + \frac{1}{2} \int_B^{t_j^S} (t_i - \bar{b}) f(t_j) dt_j \\ & \geq \int_0^{t_j^S} (t_i - B) f(t_j) dt_j + \frac{1}{2} \int_{t_j^S}^{\omega} (t_i - B) f(t_j) dt_j \end{aligned} \quad (2)$$

must hold. For all $t_i \in (t_i^S, \omega]$,

$$\begin{aligned} & \int_0^{t_j^S} (t_i - B) f(t_j) dt_j + \frac{1}{2} \int_{t_j^S}^{\omega} (t_i - B) f(t_j) dt_j \\ & \geq \int_0^{\underline{t}} (t_i - t_j) f(t_j) dt_j + \int_{\underline{t}}^{\bar{t}} (t_i - \bar{t}) f(t_j) dt_j + \int_{\bar{t}}^{\bar{b}} (t_i - t_j) f(t_j) dt_j + \frac{1}{2} \int_B^{t_j^S} (t_i - \bar{b}) f(t_j) dt_j \end{aligned} \quad (3)$$

must hold. Thus, the systems of inequalities (2) and (3) are satisfied if and only if

$$\begin{aligned} & \int_0^{\underline{t}} (t_i^S - t_j) f(t_j) dt_j + \int_{\underline{t}}^{\bar{t}} (t_i^S - \bar{t}) f(t_j) dt_j + \int_{\bar{t}}^{\bar{b}} (t_i^S - t_j) f(t_j) dt_j + \frac{1}{2} \int_B^{t_j^S} (t_i^S - \bar{b}) f(t_j) dt_j \\ & = \int_0^{t_j^S} (t_i^S - B) f(t_j) dt_j + \frac{1}{2} \int_{t_j^S}^{\omega} (t_i^S - B) f(t_j) dt_j \end{aligned} \quad (4)$$

holds.

Next, consider incentive constraints of bidder j . For all $t_j \in [0, \underline{t}]$, he reduces his expected payoff by submitting a bid that is greater or less than his type t_j . For all $t_j \in (\underline{t}, \bar{t})$, he obtains at most the same expected payoff by submitting other bids. For all $t_j \in [\bar{t}, \bar{b}]$, he reduces his

expected payoff by submitting a bid that is greater or less than his type t_j . For all $t_j \in [B, t_j^S]$,

$$\begin{aligned} & \int_0^{\underline{t}} (t_j - t_i) f(t_i) dt_i + \int_{\underline{t}}^{\bar{t}} (t_j - \underline{t}) f(t_i) dt_i + \int_{\bar{t}}^{\bar{b}} (t_j - t_i) f(t_i) dt_i + \frac{1}{2} \int_B^{t_i^S} (t_j - \bar{b}) f(t_i) dt_i \\ & \geq \int_0^{t_i^S} (t_j - B) f(t_i) dt_i + \frac{1}{2} \int_{t_i^S}^{\omega} (t_j - B) f(t_i) dt_i \end{aligned} \quad (5)$$

must hold. For all $t_j \in (t_j^S, \omega]$,

$$\begin{aligned} & \int_0^{t_i^S} (t_j - B) f(t_i) dt_i + \frac{1}{2} \int_{t_i^S}^{\omega} (t_j - B) f(t_i) dt_i \\ & \geq \int_0^{\underline{t}} (t_j - t_i) f(t_i) dt_i + \int_{\underline{t}}^{\bar{t}} (t_j - \underline{t}) f(t_i) dt_i + \int_{\bar{t}}^{\bar{b}} (t_j - t_i) f(t_i) dt_i + \frac{1}{2} \int_B^{t_i^S} (t_j - \bar{b}) f(t_i) dt_i \end{aligned} \quad (6)$$

must hold. Thus, the systems of inequalities (5) and (6) are satisfied if and only if

$$\begin{aligned} & \int_0^{\underline{t}} (t_j^S - t_i) f(t_i) dt_i + \int_{\underline{t}}^{\bar{t}} (t_j^S - \underline{t}) f(t_i) dt_i + \int_{\bar{t}}^{\bar{b}} (t_j^S - t_i) f(t_i) dt_i + \frac{1}{2} \int_B^{t_i^S} (t_j^S - \bar{b}) f(t_i) dt_i \\ & = \int_0^{t_i^S} (t_j^S - B) f(t_i) dt_i + \frac{1}{2} \int_{t_i^S}^{\omega} (t_j^S - B) f(t_i) dt_i \end{aligned} \quad (7)$$

holds. Calculating (4) and (7), respectively, we have

$$t_i^S = \frac{\omega B - (\bar{t} - \underline{t})^2}{\omega - B}$$

and

$$t_j^S = \frac{\omega B + (\bar{t} - \underline{t})^2}{\omega - B}.$$

Thus, the strategy profile σ^S is an equilibrium.

Step 2. We show that it is enough to consider the equilibrium σ^\sharp .

The strategy profile σ^S is an equilibrium. And the seller obtains the equilibrium revenue,

which is given by

$$\begin{aligned}
& \int_0^{\underline{t}} \left(\int_{t_i}^{t_j^S} t_i f(t_j) dt_j \right) f(t_i) dt_i + \int_{\underline{t}}^{\bar{t}} \left(\int_{\underline{t}}^{t_j^S} \underline{t} f(t_j) dt_j \right) f(t_i) dt_i + \int_{\underline{t}}^{\bar{b}} \left(\int_{t_i}^{t_j^S} t_i f(t_j) dt_j \right) f(t_i) dt_i \\
& + \int_B^{t_i^S} \left(\int_B^{t_j^S} \bar{b} f(t_j) dt_j \right) f(t_i) dt_i + \int_0^\omega \left(\int_{t_j^S}^\omega B f(t_j) dt_j \right) f(t_i) dt_i + \int_0^{\underline{t}} \left(\int_{t_j}^{t_i^S} t_j f(t_i) dt_i \right) f(t_j) dt_j \\
& + \int_{\underline{t}}^{\bar{t}} \left(\int_{\bar{t}}^{t_i^S} \bar{t} f(t_i) dt_i \right) f(t_j) dt_j + \int_{\bar{t}}^{\bar{b}} \left(\int_{t_j}^{t_i^S} t_j f(t_i) dt_i \right) f(t_j) dt_j + \int_0^{t_j^S} \left(\int_{t_i^S}^\omega B f(t_i) dt_i \right) f(t_j) dt_j \\
& = \frac{1}{6\omega^2} \{ 2\underline{t}^3 + 3(t_i^S - t_j^S)\underline{t}^2 - 6(t_i^S - t_j^S)\underline{t}\bar{t} - 6\underline{t}^2\bar{t} + 6\underline{t}\bar{t}^2 + 3(t_i^S - t_j^S)\bar{t}^2 - 2\bar{t}^3 \\
& \quad + 3\bar{b}^2 t_i^S - 6\bar{b} t_i^S B + 6\bar{b} t_i^S t_j^S + 3\bar{b}^2 t_j^S - 6\bar{b} t_j^S B - 6t_i^S t_j^S B - 4\bar{b}^3 + 6\bar{b} B^2 + 6\omega^2 B \}.
\end{aligned}$$

Here let $E[R^S(\underline{t}, \bar{t})] := \frac{1}{6\omega^2} \{ 2\underline{t}^3 + 3(t_i^S - t_j^S)\underline{t}^2 - 6(t_i^S - t_j^S)\underline{t}\bar{t} - 6\underline{t}^2\bar{t} + 6\underline{t}\bar{t}^2 + 3(t_i^S - t_j^S)\bar{t}^2 - 2\bar{t}^3 + 3\bar{b}^2 t_i^S - 6\bar{b} t_i^S B + 6\bar{b} t_i^S t_j^S + 3\bar{b}^2 t_j^S - 6\bar{b} t_j^S B - 6t_i^S t_j^S B - 4\bar{b}^3 + 6\bar{b} B^2 + 6\omega^2 B \}$. The equilibrium revenue $E[R^S(\cdot, \cdot)]$ is a function of the types \underline{t} and \bar{t} on the domain $[0, \bar{b}] \times [0, \bar{b}]$. Evaluating \bar{b} at B , we have

$$\frac{1}{6\omega^2} \{ 2\underline{t}^3 + 3(t_i^S - t_j^S)\underline{t}^2 - 6(t_i^S - t_j^S)\underline{t}\bar{t} - 6\underline{t}^2\bar{t} + 6\underline{t}\bar{t}^2 + 3(t_i^S - t_j^S)\bar{t}^2 - 2\bar{t}^3 + 2B^3 - 3(t_i^b + t_j^b)B^2 + 6\omega^2 B \}.$$

Partially differentiating $E[R^S(\cdot, \cdot)]$ with respect to \bar{t} , we have

$$\begin{aligned}
\frac{\partial E[R^S(\underline{t}, \bar{t})]}{\partial \bar{t}} &= \frac{1}{2\omega^2} (2\underline{t}t_j^S + 2\underline{t}\bar{t}(t_j^S)'_{\bar{t}} - \underline{t}^2(t_j^S)'_{\bar{t}} - 2\underline{t}^2 - 2\underline{t}t_j^S - \bar{t}^2(t_j^S)'_{\bar{t}} - (t_j^S)'_{\bar{t}}B^2 \\
&\quad - (t_i^S)'_{\bar{t}}B^2 + \underline{t}^2(t_i^S)'_{\bar{t}} + 2\underline{t}t_i^S + \bar{t}^2(t_i^S)'_{\bar{t}} - 2\underline{t}t_i^S - 2\underline{t}\bar{t}(t_i^S)'_{\bar{t}} - 2\bar{t}^2 + 4\underline{t}\bar{t}).
\end{aligned} \tag{8}$$

Here let $(t_i^S)'_{\bar{t}}$ and $(t_j^S)'_{\bar{t}}$ be a partial derivative of t_i^S with respect to \bar{t} and a partial derivative of t_j^S with respect to \bar{t} , respectively. Substituting

$$(t_i^S)'_{\bar{t}} = -\frac{2(\bar{t} - \underline{t})}{\omega - B}$$

and

$$(t_j^S)'_{\bar{t}} = \frac{2(\bar{t} - \underline{t})}{\omega - B}$$

into (8), we have

$$\frac{\partial E[R^S(\underline{t}, \bar{t})]}{\partial \bar{t}} = -\frac{(\bar{t} - \underline{t})^2}{\omega^2(\omega - B)} \{ 4(\bar{t} - \underline{t}) + (\omega - B) \} < 0.$$

Similarly, partially differentiating $E[R^S(\cdot, \cdot)]$ with respect to \underline{t} , we have

$$\begin{aligned} \frac{\partial E[R^S(\underline{t}, \bar{t})]}{\partial \underline{t}} &= \frac{1}{2\omega^2} (2\bar{t}t_j^S + 2\underline{t}\bar{t}(t_j^S)'_{\underline{t}} - 2\underline{t}t_j^S - \underline{t}^2(t_j^S)'_{\underline{t}} - 4\underline{t}\bar{t} + 2\underline{t}^2 - \bar{t}^2(t_j^S)'_{\underline{t}} - (t_j^S)'_{\underline{t}}B^2 \\ &\quad - (t_i^S)'_{\underline{t}}B^2 + 2\underline{t}t_i^S + \underline{t}^2(t_i^S)'_{\underline{t}} + \bar{t}^2(t_i^S)'_{\underline{t}} - 2\bar{t}t_i^S - 2\underline{t}\bar{t}(t_i^S)'_{\underline{t}} + 2\bar{t}^2). \end{aligned} \quad (9)$$

Here let $(t_i^S)'_{\underline{t}}$ and $(t_j^S)'_{\underline{t}}$ be a partial derivative of t_i^S with respect to \underline{t} and a partial derivative of t_j^S with respect to \underline{t} , respectively.

Substituting

$$(t_i^S)'_{\underline{t}} = \frac{2(\bar{t} - \underline{t})}{\omega - B}$$

and

$$(t_j^S)'_{\underline{t}} = -\frac{2(\bar{t} - \underline{t})}{\omega - B}$$

into (9), we have

$$\frac{\partial E[R^S(\underline{t}, \bar{t})]}{\partial \underline{t}} = \frac{(\bar{t} - \underline{t})^2}{\omega^2(\omega - B)} \{4(\bar{t} - \underline{t}) + (\omega - B)\} > 0.$$

From the above argument, $E[R^S(\cdot, \cdot)]$ is minimized at $(\underline{t}, \bar{t}) = (0, \bar{b})$ and maximized at $(\underline{t}, \bar{t}) = (t, t)$ for all $t \in [0, \bar{b}]$. These facts imply that it is enough to consider the equilibrium σ^\sharp .

Step 3. We derive the domain of the equilibrium revenue $E[R^\sharp(b)]$.

We can rewrite both thresholds t_i^S and t_j^S for the equilibrium σ^\sharp as follows.

$$t_i^\sharp := \frac{\omega B - b^2}{\omega - B}$$

and

$$t_j^\sharp := \frac{\omega B + b^2}{\omega - B}.$$

We need to check that both thresholds t_i^\sharp and t_j^\sharp are well-defined for each $b \in [0, \bar{b}]$. First we consider the threshold t_i^\sharp . If $b = \bar{b}$, we have

$$t_i^\sharp = \frac{\omega B - \bar{b}^2}{\omega - B}.$$

Evaluating \bar{b} at B , we have

$$t_i^\# = B.$$

As the threshold $t_i^\#$ is decreasing with respect to the type b , the threshold $t_i^\#$ is always greater than or equal to the buy price B for each $b \in [0, \bar{b}]$. If $b = 0$, on the other hand, we have

$$t_i^\# = \frac{\omega B}{\omega - B},$$

which is less than the type ω . Thus, the threshold $t_i^\#$ is always less than the type ω for each $b \in [0, \bar{b}]$.

Now, we consider the threshold $t_j^\#$. To be less than or equal to the type ω ,

$$t_j^\# = \frac{\omega B + b^2}{\omega - B} \leq \omega \iff b \leq \sqrt{\omega^2 - 2\omega B}$$

must hold. Note that $b \geq 0$. Moreover, we need to check that $\bar{b} \leq \sqrt{\omega^2 - 2\omega B}$. Raising both sides to the second power, we have

$$\bar{b}^2 \leq \omega^2 - 2\omega B.$$

Evaluating \bar{b} at B , we have

$$B \leq (\sqrt{2} - 1)\omega.$$

Thus, we can choose the $b \in [0, \bar{b}]$ arbitrarily if $B \leq (\sqrt{2} - 1)\omega$. And we can only choose the type b from the interval $[0, \sqrt{\omega^2 - 2\omega B}]$ if $B > (\sqrt{2} - 1)\omega$.

Proof of Proposition 4

We organize the proof in two steps.

Step 1. We show that a certain strategy profile is an equilibrium.

Fix $0 \leq \underline{t} \leq \bar{t} \leq \bar{b}$ arbitrarily. Then, consider the strategy profile $\sigma^N = (\sigma_i^N(\cdot), \sigma_j^N(\cdot))$ such that

$$\sigma_i^N(t_i) = \begin{cases} t_i & \text{if } 0 \leq t_i \leq \underline{t} \\ \underline{t} & \text{if } \underline{t} < t_i < \bar{t} \\ t_i & \text{if } \bar{t} \leq t_i \leq \bar{b} \\ \bar{b} & \text{if } B \leq t_i \leq \omega \end{cases}$$

and

$$\sigma_j^N(t_j) = \begin{cases} t_j & \text{if } 0 \leq t_j \leq \underline{t} \\ \bar{t} & \text{if } \underline{t} < t_j < \bar{t} \\ t_j & \text{if } \bar{t} \leq t_j \leq \bar{b} \\ \bar{b} & \text{if } B \leq t_j \leq \omega. \end{cases}$$

We show that the strategy profile σ^N is an equilibrium. Consider incentive constraints of bidder i . For all $t_i \in [0, \underline{t}]$, he reduces his expected payoff by submitting a bid that is greater or less than his type t_i . For all $t_i \in (\underline{t}, \bar{t})$, he reduces his expected payoff by submitting a bid that is greater than the bid \bar{t} . For all $t_i \in [\bar{t}, \bar{b}]$, he reduces his expected payoff by submitting a bid that is greater or less than his type t_i .

For all $t_i \in [B, \omega]$, we need to consider what the most profitable deviation is. Here let

$$\begin{aligned} \gamma_i(t_i) &:= \int_0^\omega (t_i - B)f(t_j)dt_j - \lim_{\varepsilon \rightarrow 0} \left\{ \int_0^{\underline{t}} (t_i - t_j)f(t_j)dt_j + \int_{\underline{t}}^{\bar{t}} (t_i - \bar{t})f(t_j)dt_j + \int_{\bar{t}}^{\bar{b}-\varepsilon} (t_i - t_j)f(t_j)dt_j \right\} \\ &= \frac{1}{\omega} \left\{ (\omega - B)t_i - \omega B + \frac{B^2}{2} + \frac{(\bar{t} - \underline{t})^2}{2} \right\}. \end{aligned} \tag{10}$$

The first term in the first line of (10) is the expected payoff obtained by submitting the buy price B . The second term in the first line of (10) is the expected payoff obtained by submitting a bid which is less than the bid \bar{b} . Note that we evaluate \bar{b} at B . For $t_i \in [B, \omega]$, the function $\gamma_i(\cdot)$ is increasing with respect to t_i .

Next, consider incentive constraints of bidder j . For all $t_j \in [0, \underline{t}]$, he reduces his expected payoff by submitting a bid that is greater or less than his type t_j . For all $t_j \in (\underline{t}, \bar{t})$, he reduces his expected payoff by submitting a bid that is less than the bid \underline{t} . For all $t_j \in [\bar{t}, \bar{b}]$, he reduces his expected payoff by submitting a bid that is greater or less than his type t_j .

For all $t_i \in [B, \omega]$, we need to consider what the most profitable deviation is.

$$\begin{aligned} \gamma_j(t_j) &:= \int_0^\omega (t_j - B)f(t_i)dt_i - \lim_{\varepsilon \rightarrow 0} \left\{ \int_0^{\underline{t}} (t_j - t_i)f(t_i)dt_i + \int_{\underline{t}}^{\bar{t}} (t_j - \underline{t})f(t_i)dt_i + \int_{\bar{t}}^{\bar{b}-\varepsilon} (t_j - t_i)f(t_i)dt_i \right\} \\ &= \frac{1}{\omega} \left\{ (\omega - B)t_j - \omega B + \frac{B^2}{2} - \frac{(\bar{t} - \underline{t})^2}{2} \right\}. \end{aligned} \tag{11}$$

The first term in the first line of (11) is the expected payoff obtained by submitting the buy price B . The second term in the first line of (11) is the expected payoff obtained by submitting a bid which is less than the bid \bar{b} . Note that we evaluate \bar{b} at B . For $t_j \in [B, \omega]$, the function $\gamma_j(\cdot)$ is increasing with respect to t_j . And $\gamma_j(B) < 0$.

We need to consider the following four cases.

Case 1: $\gamma_i(\omega) < 0$ and $\gamma_j(\omega) < 0$.

Calculating $\gamma_i(\omega) < 0$ and $\gamma_j(\omega) < 0$, we have

$$2\omega - \sqrt{2\omega^2 - (\bar{t} - \underline{t})^2} < B.$$

For all $t_i \in [B, \omega]$,

$$\begin{aligned} & \int_0^{\underline{t}} (t_i - t_j) f(t_j) dt_j + \int_{\underline{t}}^{\bar{t}} (t_i - \bar{t}) f(t_j) dt_j + \int_{\bar{t}}^{\bar{b}} (t_i - t_j) f(t_j) dt_j + \frac{1}{2} \int_B^{\omega} (t_i - \bar{b}) f(t_j) dt_j \\ & \geq \int_0^{\underline{t}} (t_i - t_j) f(t_j) dt_j + \int_{\underline{t}}^{\bar{t}} (t_i - \bar{t}) f(t_j) dt_j + \int_{\bar{t}}^{\bar{b}} (t_i - t_j) f(t_j) dt_j \end{aligned} \quad (12)$$

and for all $t_j \in [B, \omega]$,

$$\begin{aligned} & \int_0^{\underline{t}} (t_j - t_i) f(t_i) dt_i + \int_{\underline{t}}^{\bar{t}} (t_j - \underline{t}) f(t_i) dt_i + \int_{\bar{t}}^{\bar{b}} (t_j - t_i) f(t_i) dt_i + \frac{1}{2} \int_B^{\omega} (t_j - \bar{b}) f(t_i) dt_i \\ & \geq \int_0^{\underline{t}} (t_j - t_i) f(t_i) dt_i + \int_{\underline{t}}^{\bar{t}} (t_j - \bar{t}) f(t_i) dt_i + \int_{\bar{t}}^{\bar{b}} (t_j - t_i) f(t_i) dt_i \end{aligned} \quad (13)$$

must hold. Clearly, both the systems of inequalities (12) and (13) hold. Thus, the strategy profile σ^N is an equilibrium in second-price sealed-bid auctions with a buy price $B \in [2\omega - \sqrt{2\omega^2 - (\bar{t} - \underline{t})^2}, \omega]$.

Case 2: $\gamma_i(\omega) \geq 0$ and $\gamma_j(\omega) < 0$.

Calculating $\gamma_i(\omega) \geq 0$ and $\gamma_j(\omega) < 0$, we have

$$2\omega - \sqrt{2\omega^2 + (\bar{t} - \underline{t})^2} < B \leq 2\omega - \sqrt{2\omega^2 - (\bar{t} - \underline{t})^2}.$$

For all $t_i \in [B, t_i^N]$,

$$\begin{aligned} & \int_0^{\underline{t}} (t_i - t_j) f(t_j) dt_j + \int_{\underline{t}}^{\bar{t}} (t_i - \bar{t}) f(t_j) dt_j + \int_{\bar{t}}^{\bar{b}} (t_i - t_j) f(t_j) dt_j + \frac{1}{2} \int_B^{\omega} (t_i - \bar{b}) f(t_j) dt_j \\ & \geq \int_0^{\underline{t}} (t_i - t_j) f(t_j) dt_j + \int_{\underline{t}}^{\bar{t}} (t_i - \bar{t}) f(t_j) dt_j + \int_{\bar{t}}^{\bar{b}} (t_i - t_j) f(t_j) dt_j \end{aligned} \quad (14)$$

and for all $t_i \in (t_i^N, \omega]$,

$$\begin{aligned} & \int_0^{\underline{t}} (t_i - t_j) f(t_j) dt_j + \int_{\underline{t}}^{\bar{t}} (t_i - \bar{t}) f(t_j) dt_j + \int_{\bar{t}}^{\bar{b}} (t_i - t_j) f(t_j) dt_j + \frac{1}{2} \int_B^\omega (t_i - \bar{b}) f(t_j) dt_j \\ & \geq \int_0^\omega (t_i - B) f(t_j) dt_j \end{aligned} \quad (15)$$

and for all $t_j \in [B, \omega]$,

$$\begin{aligned} & \int_0^{\underline{t}} (t_j - t_i) f(t_i) dt_i + \int_{\underline{t}}^{\bar{t}} (t_j - \underline{t}) f(t_i) dt_i + \int_{\bar{t}}^{\bar{b}} (t_j - t_i) f(t_i) dt_i + \frac{1}{2} \int_B^\omega (t_j - \bar{b}) f(t_i) dt_i \\ & \geq \int_0^{\underline{t}} (t_j - t_i) f(t_i) dt_i + \int_{\underline{t}}^{\bar{t}} (t_j - \bar{t}) f(t_i) dt_i + \int_{\bar{t}}^{\bar{b}} (t_j - t_i) f(t_i) dt_i \end{aligned} \quad (16)$$

must hold. Here let $t_i^N := \{2\omega B - B^2 - (\bar{t} - \underline{t})^2\}/2(\omega - B)$. Clearly, both the systems of inequalities (14) and (16) hold. Thus, it is sufficient to consider whether the system of inequalities (15) holds or not.

Calculating (15), we have

$$\frac{1}{2\omega} \{-(\omega + B - 2\bar{b})t_i - (\bar{t} - \underline{t})^2 - \bar{b}^2 - \bar{b}\omega + \bar{b}B + 2\omega B\} \geq 0.$$

Evaluating \bar{b} at B , we have

$$\frac{1}{2\omega} \{-(\omega - B)t_i - (\bar{t} - \underline{t})^2 + \omega B\} \geq 0.$$

Here let

$$\iota_i(t_i) := -(\omega - B)t_i - (\bar{t} - \underline{t})^2 + \omega B.$$

For $t_i \in (t_i^N, \omega]$, $\iota_i(\cdot)$ is decreasing with respect to t_i . Thus, it is sufficient to show that (15) holds for $t_i = \omega$. That is,

$$B \geq \frac{(\bar{t} - \underline{t})^2 + \omega^2}{2\omega}.$$

Here we consider whether t_i^N is well-defined.

$$\begin{aligned} t_i^N - B &= \frac{2\omega B - B^2 - (\bar{t} - \underline{t})^2}{2(\omega - B)} - B \\ &= \frac{B^2 - (\bar{t} - \underline{t})^2}{2(\omega - B)} \geq 0. \end{aligned}$$

Thus, $t_i^N \geq B$. Next, we consider whether

$$t_i^N = \frac{2\omega B - B^2 - (\bar{t} - \underline{t})^2}{2(\omega - B)} \leq \omega. \quad (17)$$

Calculating (17), we have

$$B \leq 2\omega - \sqrt{2\omega^2 - (\bar{t} - \underline{t})^2}.$$

Thus, $t_i^N \leq \omega$.

To summarize, the strategy profile σ^N is an equilibrium in second-price sealed-bid auctions with a buy price $B \in [2\omega - \sqrt{2\omega^2 + (\bar{t} - \underline{t})^2}, 2\omega - \sqrt{2\omega^2 - (\bar{t} - \underline{t})^2}]$ when $2\omega - \sqrt{2\omega^2 + (\bar{t} - \underline{t})^2} > \frac{(\bar{t} - \underline{t})^2 + \omega^2}{2\omega}$. And the strategy profile σ^N is an equilibrium in second-price sealed-bid auctions with a buy price $B \in [\frac{(\bar{t} - \underline{t})^2 + \omega^2}{2\omega}, 2\omega - \sqrt{2\omega^2 - (\bar{t} - \underline{t})^2}]$ when $2\omega - \sqrt{2\omega^2 + (\bar{t} - \underline{t})^2} \leq \frac{(\bar{t} - \underline{t})^2 + \omega^2}{2\omega}$.

Case 3: $\gamma_i(\omega) < 0$ and $\gamma_j(\omega) \geq 0$.

There is no buy price B such that $\gamma_i(\omega) < 0$ and $\gamma_j(\omega) \geq 0$ hold simultaneously.

Case 4: $\gamma_i(\omega) \geq 0$ and $\gamma_j(\omega) \geq 0$.

Calculating $\gamma_i(\omega) \geq 0$ and $\gamma_j(\omega) \geq 0$, we have

$$B \leq 2\omega - \sqrt{2\omega^2 + (\bar{t} - \underline{t})^2}.$$

For all $t_i \in [B, t_i^N]$,

$$\begin{aligned} & \int_0^{\underline{t}} (t_i - t_j) f(t_j) dt_j + \int_{\underline{t}}^{\bar{t}} (t_i - \bar{t}) f(t_j) dt_j + \int_{\bar{t}}^{\bar{b}} (t_i - t_j) f(t_j) dt_j + \frac{1}{2} \int_B^{\omega} (t_i - \bar{b}) f(t_j) dt_j \\ & \geq \int_0^{\underline{t}} (t_i - t_j) f(t_j) dt_j + \int_{\underline{t}}^{\bar{t}} (t_i - \bar{t}) f(t_j) dt_j + \int_{\bar{t}}^{\bar{b}} (t_i - t_j) f(t_j) dt_j \end{aligned} \quad (18)$$

and for all $t_i \in (t_i^N, \omega]$,

$$\begin{aligned} & \int_0^{\underline{t}} (t_i - t_j) f(t_j) dt_j + \int_{\underline{t}}^{\bar{t}} (t_i - \bar{t}) f(t_j) dt_j + \int_{\bar{t}}^{\bar{b}} (t_i - t_j) f(t_j) dt_j + \frac{1}{2} \int_B^{\omega} (t_i - \bar{b}) f(t_j) dt_j \\ & \geq \int_0^{\omega} (t_i - B) f(t_j) dt_j \end{aligned} \quad (19)$$

and for all $t_j \in [B, t_j^N]$,

$$\begin{aligned} & \int_0^{\underline{t}} (t_j - t_i) f(t_i) dt_i + \int_{\underline{t}}^{\bar{t}} (t_j - \underline{t}) f(t_i) dt_i + \int_{\bar{t}}^{\bar{b}} (t_j - t_i) f(t_i) dt_i + \frac{1}{2} \int_B^{\omega} (t_j - \bar{b}) f(t_i) dt_i \\ & \geq \int_0^{\underline{t}} (t_j - t_i) f(t_i) dt_i + \int_{\underline{t}}^{\bar{t}} (t_j - \bar{t}) f(t_i) dt_i + \int_{\bar{t}}^{\bar{b}} (t_j - t_i) f(t_i) dt_i \end{aligned} \quad (20)$$

and for all $t_j \in (t_j^N, \omega]$,

$$\int_0^{\underline{t}} (t_j - t_i) f(t_i) dt_i + \int_{\underline{t}}^{\bar{t}} (t_j - \underline{t}) f(t_i) dt_i + \int_{\bar{t}}^{\bar{b}} (t_j - t_i) f(t_i) dt_i + \frac{1}{2} \int_B^{\omega} (t_j - \bar{b}) f(t_i) dt_i \geq \int_0^{\omega} (t_j - B) f(t_i) dt_i \quad (21)$$

must hold. Clearly, both the systems of inequalities (18) and (20) hold. Thus, it is sufficient to consider whether the systems of inequalities (19) and (21) hold or not.

Calculating (21), we have

$$\frac{1}{2\omega} \{ -(\omega + B - 2\bar{b})t_j + (\bar{t} - \underline{t})^2 - \bar{b}^2 - \bar{b}\omega + \bar{b}B + 2\omega B \} \geq 0.$$

Evaluating \bar{b} at B , we have

$$\frac{1}{2\omega} \{ -(\omega - B)t_j + (\bar{t} - \underline{t})^2 + \omega B \} \geq 0.$$

Here let

$$\iota_j(t_j) := -(\omega - B)t_j + (\bar{t} - \underline{t})^2 + \omega B.$$

For $t_j \in [B, \omega]$, $\iota_j(\cdot)$ is decreasing with respect to t_j . Thus, it is sufficient to show that (21) holds for $t_j = \omega$. That is,

$$B \geq \frac{-(\bar{t} - \underline{t})^2 + \omega^2}{2\omega}.$$

To summarize, the strategy profile σ^N is an equilibrium in second-price sealed-bid auctions with a buy price $B \in [\frac{(\bar{t} - \underline{t})^2 + \omega^2}{2\omega}, 2\omega - \sqrt{2\omega^2 + (\bar{t} - \underline{t})^2}]$ when $2\omega - \sqrt{2\omega^2 + (\bar{t} - \underline{t})^2} > \frac{(\bar{t} - \underline{t})^2 + \omega^2}{2\omega}$. And the strategy profile σ^N is not an equilibrium when $2\omega - \sqrt{2\omega^2 + (\bar{t} - \underline{t})^2} \leq \frac{(\bar{t} - \underline{t})^2 + \omega^2}{2\omega}$.

From the above argument, the strategy profile σ^N is an equilibrium in second-price sealed-bid auctions with a buy price $B \in [\frac{(\bar{t} - \underline{t})^2 + \omega^2}{2\omega}, \omega]$.

Step 2. We show that it is enough to consider the equilibrium σ^\natural .

The strategy profile σ^N is an equilibrium. And the seller receives the equilibrium revenue, which is given by

$$\begin{aligned} & \int_0^{\underline{t}} \left(\int_{t_i}^{\omega} t_i f(t_j) dt_j \right) f(t_i) dt_i + \int_{\underline{t}}^{\bar{t}} \left(\int_{\underline{t}}^{\omega} \underline{t} f(t_j) dt_j \right) f(t_i) dt_i + \int_{\bar{t}}^{\bar{b}} \left(\int_{t_i}^{\omega} t_i f(t_j) dt_j \right) f(t_i) dt_i \\ & + \int_0^{\underline{t}} \left(\int_{t_j}^{\omega} t_j f(t_i) dt_i \right) f(t_j) dt_j + \int_{\underline{t}}^{\bar{t}} \left(\int_{\bar{t}}^{\omega} \bar{t} f(t_i) dt_i \right) f(t_j) dt_j + \int_{\bar{t}}^{\bar{b}} \left(\int_{t_j}^{\omega} t_j f(t_i) dt_i \right) f(t_j) dt_j \\ & + \int_B^{\omega} \left(\int_B^{\omega} \bar{b} f(t_i) dt_i \right) f(t_j) dt_j \\ & = \frac{1}{3\omega^2} \{ -2\bar{b}^3 + 3\bar{b}^2\omega + 3\bar{b}\omega^2 - 6\omega\bar{b}B + 3\bar{b}B^2 - (\bar{t} - \underline{t})^3 \}. \end{aligned}$$

Here let $E[R^N(t, \bar{t})] := \frac{1}{3\omega^2} \{-2\bar{b}^3 + 3\bar{b}^2\omega + 3\bar{b}\omega^2 - 6\omega\bar{b}B + 3\bar{b}B^2 - (\bar{t} - \underline{t})^3\}$. The equilibrium revenue $E[R^N(\cdot, \cdot)]$ is a function of the types \underline{t} and \bar{t} on the domain $[0, \bar{b}] \times [0, \bar{b}]$. Evaluating \bar{b} at B , we have

$$\frac{1}{3\omega^2} \{3\omega^2 B - 3\omega B^2 + B^3 - (\bar{t} - \underline{t})^3\}.$$

Clearly, $E[R^N(\cdot, \cdot)]$ is decreasing with respect to \bar{t} , and increasing with respect to \underline{t} . Thus, $E[R^N(\cdot, \cdot)]$ is minimized at $(\underline{t}, \bar{t}) = (0, \bar{b})$ and maximized at $(\underline{t}, \bar{t}) = (t, t)$ for all $t \in [0, \bar{b}]$. These facts imply that it is enough to consider the equilibrium σ^{\sharp} .

Proof of Theorem 1

By Propositions 3 and 4, the minimum of equilibrium revenues of second-price sealed-bid auctions with a buy price $B \in [0, \omega]$ is

$$\begin{aligned} E[R^{\sharp}(\bar{b})] & \text{ if } 0 < B \leq (\sqrt{2} - 1)\omega \\ E[R^{\sharp}(\hat{b})] & \text{ if } (\sqrt{2} - 1)\omega < B \leq \frac{\omega}{2} \\ E[R^{\sharp}(\tilde{b})] & \text{ if } \frac{\omega}{2} < B \leq \omega. \end{aligned}$$

At first, $E[R^{\sharp}(\bar{b})] \leq E[R^{\sharp}(\hat{b})]$ for any buy price $B \in (0, \frac{\omega}{2}]$. Thus, we consider $E[R^{\sharp}(\bar{b})]$ instead of $E[R^{\sharp}(\hat{b})]$ for a buy price $B \in ((\sqrt{2} - 1)\omega, \frac{\omega}{2}]$. Calculating $E[R^{\sharp}(\bar{b})]$, we have

$$E[R^{\sharp}(\bar{b})] = \frac{B}{\omega^2(\omega - B)}(\omega^3 - \omega^2 B - \omega B^2 - B^3).$$

Note that we evaluate \bar{b} at B . $\frac{B}{\omega^2(\omega - B)}$ is positive for any buy price $B \in (0, \frac{\omega}{2}]$. Differentiating $\omega^3 - \omega^2 B - \omega B^2 - B^3$ with respect to B , we have

$$-3\left(B + \frac{\omega}{3}\right)^2 - \frac{2}{3}\omega^2 < 0.$$

Thus, $\omega^3 - \omega^2 B - \omega B^2 - B^3$ is minimized at $B = \frac{\omega}{2}$. In this case, we have

$$\omega^3 - \omega^2 B - \omega B^2 - B^3 = \frac{\omega^3}{8} > 0.$$

To summarize, $E[R^{\sharp}(\bar{b})]$ is positive for any buy price $B \in (0, \frac{\omega}{2}]$. When $B = (\sqrt{2} - 1)\omega$, in addition, $\bar{b} = (\sqrt{2} - 1)\omega = \hat{b}$. Thus, $E[R^{\sharp}(\bar{b})] = E[R^{\sharp}(\hat{b})]$ at the buy price $B = (\sqrt{2} - 1)\omega$.

On the other hand,

$$E[R^{\sharp}(\tilde{b})] \geq \frac{1}{\omega}(-B^2 + \omega B).$$

Note that we evaluate \bar{b} at B . Differentiating $-B^2 + \omega B$ with respect to B , we have

$$-2B + \omega < 0.$$

Thus, $\frac{1}{\omega}(-B^2 + \omega B)$ is minimized at $B = \omega$. In this case, we have

$$\frac{1}{\omega}(-B^2 + \omega B) = 0.$$

To summarize, $E[R^\sharp(\tilde{b})]$ is positive for any buy price $B \in (\frac{\omega}{2}, \omega)$. Note that $E[R^\sharp(\hat{b})] = \frac{7\omega}{24} = E[R^\sharp(\tilde{b})]$ at the buy price $B = \frac{\omega}{2}$ because $\hat{b} = 0 = \tilde{b}$.

From the above argument, a seller always obtains a positive equilibrium revenue by introducing a buy price appropriately.

The effects of a reserve price on equilibrium revenues

We reconsider second-price sealed-bid auctions without a buy price B . In these auctions, introducing a reserve price is the most straightforward way for a seller to guarantee a positive equilibrium revenue. We then consider the effects of a reserve price on equilibrium revenues.

Suppose that a seller sets a reserve price $r \in [0, \omega]$ arbitrarily. Fix $b \in [r, \omega]$ arbitrarily. Then, the strategy profile σ^b such that

$$\sigma_i^b(t_i) = \begin{cases} 0 & \text{if } 0 \leq t_i \leq r \\ r & \text{if } r < t_i < b \\ t_i & \text{if } b \leq t_i \leq \omega \end{cases}$$

and

$$\sigma_j^b(t_j) = \begin{cases} 0 & \text{if } 0 \leq t_j \leq r \\ b & \text{if } r < t_j < b \\ t_j & \text{if } b \leq t_j \leq \omega. \end{cases}$$

The strategy profile σ^b is an equilibrium. And the seller obtains the equilibrium revenue, which is given by

$$\frac{1}{3\omega^2}(\omega^3 + 3r^2\omega - 3r^3 - b^3 + 3b^2r - 3br^2).$$

Here let $E[R^r(b)] := \frac{1}{3\omega^2}(\omega^3 + 3r^2\omega - 3r^3 - b^3 + 3b^2r - 3br^2)$. The equilibrium revenue $E[R^r(\cdot)]$ is a function of the type b on the domain $[r, \omega]$.

We then have a similar result to that of Proposition 1.

Proposition 5. *Consider second-price sealed-bid auctions with a reserve price $r \in [0, \omega]$. Then, the set of equilibrium revenues is $[E[R^r(\omega)], E[R^r(r)]]$.*

Proof. Fix $r \leq \underline{t} \leq \bar{t} \leq \omega$ arbitrarily. Then, consider the following strategy profile.

$$\sigma_i^F(t_i) = \begin{cases} 0 & \text{if } 0 \leq t_i \leq r \\ t_i & \text{if } r < t_i \leq \underline{t} \\ \underline{t} & \text{if } \underline{t} < t_i < \bar{t} \\ t_i & \text{if } \bar{t} \leq t_i \leq \omega \end{cases}$$

and

$$\sigma_j^F(t_j) = \begin{cases} 0 & \text{if } 0 \leq t_j \leq r \\ t_j & \text{if } r < t_j \leq \underline{t} \\ \bar{t} & \text{if } \underline{t} < t_j < \bar{t} \\ t_j & \text{if } \bar{t} \leq t_j \leq \omega. \end{cases}$$

We show that the strategy profile σ^F is an equilibrium. Consider incentive constraints of bidder i . For all $t_i \in [0, r]$, he reduces his expected payoff by submitting a bid that is greater than the reserve price r . For all $t_i \in (r, \underline{t}]$, he reduces his expected payoff by submitting a bid that is greater or less than his type t_i . For all $t_i \in (\underline{t}, \bar{t})$, he reduces his expected payoff by submitting other bids, for example, the bid \bar{t} . For all $t_i \in [\bar{t}, \omega]$, he reduces his expected payoff by submitting a bid that is greater or less than his type t_i .

Similarly, consider incentive constraints of bidder j . For all $t_j \in [0, r]$, he reduces his expected payoff by submitting a bid that is greater than the reserve price r . For all $t_j \in (r, \underline{t}]$, he reduces his expected payoff by submitting a bid that is greater or less than his type t_j . For all $t_j \in (\underline{t}, \bar{t})$, he reduces his expected payoff by submitting other bids, for example, the bid \underline{t} . For all $t_j \in [\bar{t}, \omega]$, he reduces his expected payoff by submitting a bid that is greater or less than his type t_j .

The strategy profile σ^F is an equilibrium. And the seller obtains the equilibrium revenue, which is given by

$$\begin{aligned} & \int_0^r \left(\int_r^\omega r f(t_j) dt_j \right) f(t_i) dt_i + \int_r^{\underline{t}} \left(\int_{t_i}^\omega t_i f(t_j) dt_j \right) f(t_i) dt_i + \int_{\underline{t}}^{\bar{t}} \left(\int_{\underline{t}}^\omega \underline{t} f(t_j) dt_j \right) f(t_i) dt_i \\ & + \int_{\bar{t}}^\omega \left(\int_{t_i}^\omega t_i f(t_j) dt_j \right) f(t_i) dt_i + \int_0^r \left(\int_r^\omega r f(t_i) dt_i \right) f(t_j) dt_j + \int_r^{\underline{t}} \left(\int_{t_j}^\omega t_j f(t_i) dt_i \right) f(t_j) dt_j \\ & + \int_{\underline{t}}^{\bar{t}} \left(\int_{\bar{t}}^\omega \bar{t} f(t_i) dt_i \right) f(t_j) dt_j + \int_{\bar{t}}^\omega \left(\int_{t_j}^\omega t_j f(t_i) dt_i \right) f(t_j) dt_j \\ & = \frac{1}{3\omega^2} \{ \omega^3 + 3r^2\omega - 4r^3 - (\bar{t} - \underline{t})^3 \}. \end{aligned}$$

Here let $E[R^r(\underline{t}, \bar{t})] = \frac{1}{3\omega^2} \{ \omega^3 + 3r^2\omega - 4r^3 - (\bar{t} - \underline{t})^3 \}$. Clearly, $E[R^r(\cdot, \cdot)]$ is decreasing with respect to \bar{t} and increasing with respect to \underline{t} . Thus, $E[R^r(\cdot, \cdot)]$ is minimized at $(\underline{t}, \bar{t}) = (r, \omega)$ and maximized at $(\underline{t}, \bar{t}) = (t, t)$ for all $t \in [r, \omega]$. *Q.E.D.*

If a seller sets a reserve price r , then the equilibrium revenue is restricted. In a result, she can obtain a positive equilibrium revenue. And the seller acquires the maximum equilibrium revenue at a symmetric strategy equilibrium.

As well as the case of second-price sealed-bid auctions with a buy price B , we focus on the maximum of equilibrium revenues of each second-price sealed-bid auction with a reserve price r . Here let $ER_{\max}^r(r)$ be a function of the maximum equilibrium revenues in second-price sealed-bid auctions with a reserve price r . Then, we have

$$ER_{\max}^r(r) = \frac{1}{3\omega^2}(-4r^3 + 3\omega r^2 + \omega^3) \quad \text{if } 0 \leq r \leq \omega.$$

To illustrate, we provide an example.

Example 3. Suppose that bidders' types are distributed on the interval $[0, 1]$. Then, the maximum of equilibrium revenues is given by

$$ER_{\max}^r(r) = -\frac{4}{3}r^3 + r^2 + \frac{1}{3} \quad \text{if } 0 \leq r \leq \omega.$$

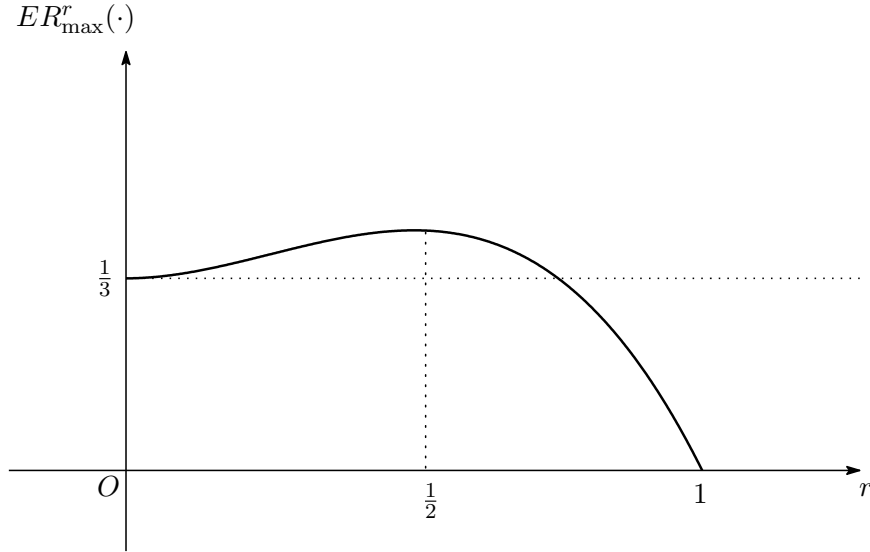


Figure 3: The maximum of equilibrium revenues $ER_{\max}^r(r)$

In Figure 3, $ER_{\max}^r(\cdot)$ is monotone increasing with respect to a reserve price $r \in [0, \frac{1}{2}]$. On the other hand, $ER_{\max}^r(\cdot)$ is monotone decreasing with respect to a reserve price $r \in (\frac{1}{2}, \omega]$.

Next, we focus on the minimum of equilibrium revenues of each second-price sealed-bid auction with a reserve price r . Here let $ER_{\min}^r(r)$ be a function of the minimum equilibrium revenues in second-price sealed-bid auctions with a reserve price r . Then, we have

$$ER_{\min}^r(r) = \frac{1}{\omega^2}(-r^3 + \omega^2 r) \quad \text{if } 0 \leq r \leq \omega.$$

We give an illustrative example.

Example 4. Suppose that bidders' types are distributed on the interval $[0, 1]$. Then, the minimum of equilibrium revenues is given by

$$ER_{\min}^r(r) = -r^3 + r \quad \text{if } 0 \leq r \leq \omega.$$

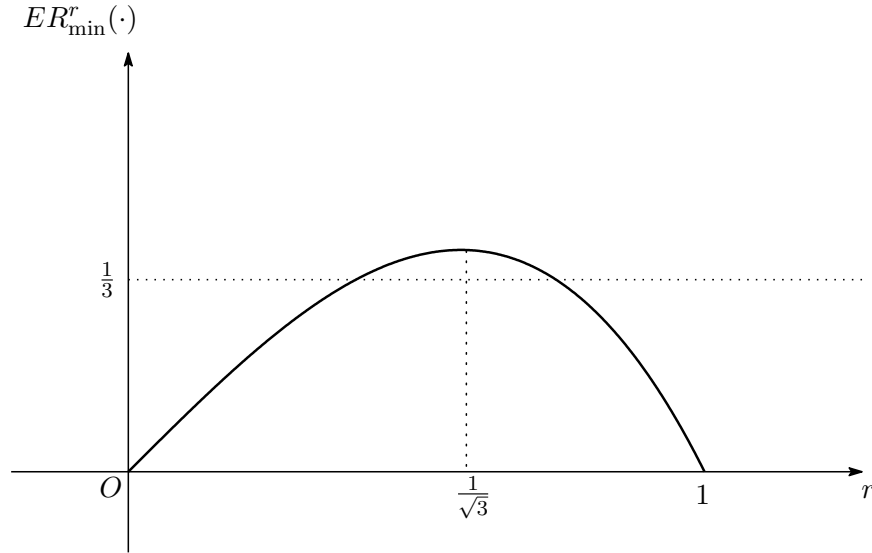


Figure 4: The minimum of equilibrium revenues $ER_{\min}^r(r)$

In Figure 4, $ER_{\min}^B(\cdot)$ is monotone increasing with respect to a reserve price $r \in [0, \frac{1}{\sqrt{3}}]$. On the other hand, $ER_{\min}^B(\cdot)$ is monotone decreasing with respect to a reserve price $r \in (\frac{1}{\sqrt{3}}, \omega]$. In Figure 4, $ER_{\min}^B(\cdot)$ is absolutely positive for any reserve price $r \in (0, 1)$.

References

- Bikhchandani, S. and Riley, J. G., 1991. "Equilibria in open common value auctions," *Journal of Economic Theory* Vol.53 (1), pp. 101–130.
- Blume, A. and Heidhues, P., 2001. "All equilibria of the Vickery auction." Working Paper. <http://www.wiwi.uni-bonn.de/heidhues/Paper/characterization2.PDF>.
- 2004. "All equilibria of the Vickery auction," *Journal of Economic Theory* Vol.114 (1), pp. 170–177.
- Budish, E. B. and Takeyama, L. N., 2001. "Buy prices in online auctions: irrationality on the internet ?" *Economics letters* Vol.72 (3), pp. 325–333.
- Che, X., 2011. "Internet auctions with a temporary buyout option," *Economics Letters* Vol.110 (3), pp. 268–271.

- Hidvégi, Z., Wang, W., and Whinston, A. B., 2006. “Buy-price English auction,” *Journal of Economic Theory* Vol.129 (1), pp. 31–56.
- Inami, Y., 2011. “The buy price in auctions with discrete type distributions,” *Mathematical Social Sciences* Vol.61 (1), pp. 1–11.
- 2018. “Which equilibrium yields the highest expected revenue in second-price sealed-bid auctions?” *Economics Letters* Vol.167, pp. 97–98.
- Lizzeri, A. and Persico, N., 2000. “Uniqueness and existence of equilibrium in auctions with a reserve price,” *Games and Economic Behavior* Vol.30 (1), pp. 83–114.
- Maskin, E. S. and Riley, J. G., 1985. “Auction theory with private values,” *American Economic Review* Vol.75 (2), pp. 150–155.
- Mathews, T., 2004. “The impact of discounting on an auction with a buyout option: a theoretical analysis motivated by eBay’s Buy-It-Now feature,” *Journal of Economics* Vol.81 (1), pp. 25–52.
- Mathews, T. and Katzman, B., 2006. “The role of varying risk attitudes in an auction with a buyout option,” *Economic Theory* Vol.27 (3), pp. 597–613.
- Milgrom, P. R., 1981. “Rational expectations, information acquisition, and competitive bidding,” *Econometrica* Vol.49 (4), pp. 921–943.
- Plum, M., 1992. “Characterization and computation of Nash-equilibria for auctions with incomplete information,” *International Journal of Game Theory* Vol.20 (4), pp. 393–418.
- Reynolds, S. S. and Wooders, J., 2009. “Auctions with a buy price,” *Economic Theory* Vol.38 (1), pp. 9–39.
- Shunda, N., 2009. “Auctions with a buy price: The case of reference-dependent preferences,” *Games and Economic Behavior* Vol.67 (2), pp. 645–664.