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Abstract

What is the optimal group size in the voluntary provision of public goods in a purely altruistic economy? The popular consensus on this fundamental question is that the free-rider problem worsens as group size increases. This study provides a counterexample of the consensus, by featuring threshold preferences that are plausible for certain typical public goods. Under these preferences, marginal utility hardly diminishes below a threshold level, but declines significantly in close proximity to the threshold, and nearly drops to zero above the threshold. We indicate that threshold preferences significantly reduce inefficiency. We also show that if marginal costs increase, threshold preferences are contrary to the consensus and lead to a partly positive relationship between efficiency and group size, thereby detecting the local efficient group size. Moreover, the local efficient group size is proportional to the slope of marginal costs as well as the threshold of marginal utility.

Keywords: threshold preferences; voluntary provision of public goods; group size

JEL classification: H41, D61

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1 Introduction

The effect of the group size of a voluntary economy on the degree of efficiency in the private provision of public goods, has been one of the fundamental questions in public economics since the pioneering work of Olson (1965). To date, many researchers have studied the issue. At least in simple homogeneous models, theoretical studies have reached a broad consensus that there is a monotonically decreasing relationship between efficiency and group size in a purely altruistic economy with noncooperative contributors; for example, refer to Mueller (1989) for Cobb–Douglas preferences and Cornes and Sandler (1985) for quasi-linear preferences, among others.[1] Isaac and Walker (1988) provide experimental support for the theoretical consensus.

Although we often take such consensus for granted, in this study we show that, in particular cases, there is more to the issue than previously understood. In early theoretical works, possible preference features of public goods have not been sufficiently addressed.[2] Recognizing that there are no appropriate preferences for all types of public goods, it is even more important to investigate how (in)efficient resource allocation can arise from every conceivable angle. Specifically, a priori argument that the diminishing degree of marginal utility depends to a greater extent on the usual consumption level, cannot be ruled out. As part of possible preferences, the present analysis considers the case of individuals with a threshold preference for public goods consumption. Under this threshold, marginal utility hardly diminishes below a minimum threshold, but declines significantly in close proximity to the threshold, nearly equaling zero above the threshold; that is, unlike the standard case, utility increases in an approximately linear fashion below the threshold and hardly increases above the threshold, as shown in Figure 1. We refer to this as individuals' threshold preference (of public goods).

[Insert Figure 1 around here]

The threshold preference is in fact considered plausible in some cases of well-recognized public goods such as volunteer work and clean environments; for example, one could imagine a charity for natural-disaster victims as a social contribution issue. In this case, the

[1] In this study, unless noted otherwise, we use "altruism" to refer to pure altruism in which individuals have a preference depending only on the total supply of public goods. While beyond the scope of this study, a branch of literature considers impure altruism (warm-glow giving) in which individuals have a preference depending on their own contributions; see Andreoni (1990) and Allouch (2010) for the details of an impure altruism case, among others. In addition, some research highlights roles such as voting, public provision, and cooperative behavior of contributors; see Cornes and Sandler (1984), Bernheim (1986), Steinberg (1987), Fehr and Gachter (2000), Slavov (2014), and Borissov *et al.* (2019), among others.

[2] One important exception is Hayashi and Ohta (2007), who consider satiation at certain consumption levels of public goods, as detailed below.

marginal utility of supportive activities could potentially decrease slightly at insufficient levels of total contributions, because the victims still tolerate the inconveniences of an unsettled lifestyle. Rather than assuming an immediate diminishing rate of the marginal utility, it is more likely that it decreases substantially only after total contributions sufficiently aid the victims. Once their standards of living are sufficiently restored, the marginal utility could become virtually zero.

As another example, consider garbage-strewn beaches and a local beach clean-up effort as an environmental issue. Irrespective of how much waste some people collect, the marginal utility of an additional clean-up would remain high as long as the remaining waste is visible, thereby spoiling an intrinsically beautiful landscape. Only after the beaches are restored to a satisfactory level of cleanliness, the marginal utility would finally begin to noticeably decrease. Eventually, little marginal utility is derived from removing the remaining limited and inconspicuous waste.

In contrast to the preceding theoretical consensus, our analysis reveals that the provision of public goods in the Nash equilibrium can lead to efficient outcomes. To this end, we consider a very simple and standard model in literature assuming an altruistic economy, except the utility function with the following three features: (a) marginal utility diminishes relatively slowly when public goods are below the threshold, (b) it declines relatively sharply only when public goods provision is in close proximity to a threshold value, and (c) marginal utility is almost zero when it exceeds the threshold. We emphasize that these three features are associated with relative changes in marginal utility and retain the standard assumptions; that is, positive marginal utility and the law of diminishing marginal utility. Thus, our analysis is consistent with the underlying framework in altruism literature, but nevertheless concludes that voluntary provision of public goods can attain efficient resource allocation.

In addition, we find that if the marginal cost of contributions from individuals increases, threshold preferences are contrary to the broad consensus that an increase in group size inevitably leads to lower efficiency. Although considerable existing literature assumes constant marginal cost, an increase in marginal cost seems rather natural in the provision of certain public goods. For example, if the provision of public goods involves specific physical tasks, contributors gradually become fatigued as their contributions increase and eventually become exhausted. The beforementioned examples of volunteering activities and keeping the environment clean, apply to this case.[3] Based on this natural assumption, we demonstrate that threshold preferences lead to a partly positive relationship between efficiency and group size, detecting the local efficient group size. Within the confines of simple noncooperative behavior of contributors in a purely altruistic economy,

[3]Other examples of increasing marginal cost can be found in the work of Hayashi and Ohta (2007).

this study is the first to uncover the local efficient group size.

This study is at the crossroads of two lines of research. One is the work raising the issue of efficiency–group size nexus in the voluntary provision of public goods in a purely altruistic economy, with certain representative studies already mentioned above. Among others, Hayashi and Ohta (2007) deserve a special mention as significant precursors to this study. The authors embraced this idea by positing two assumptions: increasing marginal cost of voluntary provision and the existence of a finite satiety point in utility. As a result, they claim that inefficiency is alleviated as group size increases, and optimality is achieved when group size approaches infinity. While their work overlaps with ours, there are notable differences in both assumptions and findings. Specifically, we assume threshold preferences instead of a finite satiety point in utility and find substantial efficiency improvements when group size is not infinite. The characteristics of threshold preferences are similar to the assumption of satiation in that the threshold level of provision could be interpreted as roughly corresponding to the satiation level. However, there are crucial differences between the two: in threshold preferences, marginal utility diminishes slowly below the threshold level and declines sharply in close proximity to the threshold level. In addition to these differences, we relax the assumption of the increasing marginal cost in which the marginal cost is required to approach zero as the contributions of individuals approach zero.

The second line of research refers to the characterization of types of public goods (e.g., pure and impure public goods). Traditionally, various types include congestible goods and local public goods. In this line of research, this study is most closely related to the recent growing body of literature on the threshold public goods (e.g., Cadsby and Maynes, 1999; Spencer *et al.*, 2009; Brekke *et al.*, 2017; Cartwright *et al.*, 2019). While plenty literature exploits experimental approaches to investigate a public goods game, threshold public goods are characterized as consumable only when total contributions surpass a minimum threshold (provision point). Thus, threshold public goods and present threshold preferences are analogous in that total contributions have meaningful threshold values. However, they critically differ from each other in that marginal utility is only positive after total contributions exceed a critical level in threshold public goods, whereas it is sizable only when below a critical level in threshold preferences.

The remainder of this paper is organized as follows. In Section 2, we present our analytical framework. We formally define threshold preferences and elucidate their properties in Section 3. In Section 4, we analytically examine the relationship between efficiency and group size, and we discuss other possible threshold preferences in Section 5. Concluding remarks are presented in Section 6.

2 Analytical framework

2.1 The model

Consider an economy in which n identical individuals exist, with $n \in (1, \infty)$. Let x and G denote a private good (numeraire) and a public good, respectively. It is considered that the individuals' preferences are represented as a quasilinear utility function

$$U = U(x, G) = x + f(G), \tag{1}$$

where $f(G)$ satisfies the standard assumptions, and is strictly increasing and concave, $f' > 0$ and $f'' < 0$. The marginal rate of substitution is given by

$$\pi(G) = \frac{\partial U/\partial G}{\partial U/\partial x} = f'(G). \tag{2}$$

A public good is supplied by the voluntary contributions of individuals, and the total contributions G are the sum of all individuals' voluntary contributions g (i.e., $G = \sum g$). An individual's endowment w is divided into two parts: private consumption x and the cost of their contribution to a public good $c(g)$. Hence, each individual faces the budget constraint

$$w = x + c(g). \tag{3}$$

As the property of cost function $c(g)$, we consider two cases of the marginal cost $c'(g)$. The first case is a standard assumption in literature, that is, the marginal cost is positive and constant (i.e., $c'(g) = \theta > 0$). In the second case, as per Hayashi and Ohta (2007), it is assumed that the marginal cost is increasing (i.e., $c''(g) > 0$). Subsequently, we assume the following conditions for each case to examine the relationship between group size and efficiency.

Assumption 1. *If $c'(g)$ is constant θ, then $0 < \theta < f'(0)$.*

Assumption 2. *If $c'(g)$ increases and $c''(g) > 0$, then $0 \leq c'(0) < f'(0)$.*

In Assumptions 1 and 2, exclusions of $\theta = f'(0)$ and $c'(0) = f'(0)$ are respectively required to guarantee that solutions are interior. Moreover, we include remarks on Assumption 2, related to Hayashi and Ohta (2007). To present a counterexample of the inverse relationship between group size and efficiency, Hayashi and Ohta (2007) impose a restriction on increasing marginal costs so that they approach zero as $g \to 0$ (i.e., $c'(0) = 0$). Note that Assumption 2 encompasses this restriction and allows for positive values (i.e., $c'(0) > 0$).

2.2 Nash equilibrium and efficient resource allocation

The present problem with Nash behavior of an individual is to maximize (1) subject to (3) by selecting x and g, given contributions by others. Noting that our model is an identical economy, the following equation is satisfied in the Nash equilibrium

$$f'\left(\overline{G}\right) = c'\left(\overline{g}\right), \tag{4}$$

where $\overline{G} = n\overline{g}$, and upper bars represent the equilibrium values. In preparation for the following analysis, it is useful to consider the group size elasticity. Total differentiation of (4) with respect to $\overline{G}$ and n yields the group size elasticity of equilibrium provision:

$$\frac{d\overline{G}}{dn}\frac{n}{\overline{G}} = \frac{c''\left(\overline{g}\right)/n}{c''\left(\overline{g}\right)/n - f''\left(\overline{G}\right)}. \tag{5}$$

Contrarily, by solving the problem to maximize (1) subject to (3) by selecting x and G, we obtain the Samuelson condition:

$$f'\left(G^*\right) = \frac{c'\left(g^*\right)}{n}, \tag{6}$$

where $G^* = ng^*$, and the asterisks represent optimal values. As before, total differentiation of (6) with respect to G^* and n yields the group size elasticity of optimal provision:

$$\frac{dG^*}{dn}\frac{n}{G^*} = \frac{c''\left(g^*\right)/n + c'\left(g^*\right)/G^*}{c''\left(g^*\right)/n - nf''\left(G^*\right)}. \tag{7}$$

3 Threshold preferences

Within the confines of the standard features of the utility function presented in the preceding section (i.e., the law of diminishing marginal utility, $f' > 0$ and $f'' < 0$), we propose threshold preferences in which the marginal utility, f', rapidly diminishes in close proximity to a threshold level of public goods γ. A crucial point to be stressed is that threshold preferences are nested in standard preferences. In other words, threshold preferences are obtained from standard preferences by assuming additional conditions of the latter. Additional conditions are imposed on the marginal utility and its first and second derivative. Specifically, threshold preferences are defined as follows.

Definition 1. *The utility function $U(x, G)$ is referred to as a threshold utility function if the marginal utility, $\partial U/\partial G = f'(G) > 0$, is a twice continuously differentiable function with $f''(G) < 0$ for all $G \in [0, \infty)$, and if it satisfies the following properties:*
(i) $\lim_{G\to\infty} f'(G) = 0$, *and* $\lim_{G\to\infty} f''(G) = 0$.

(ii) $f'(G)$ *has a unique inflection point* $(\gamma, f'(\gamma))$*; that is,* $f'''(G) = 0$ *if and only if* $G = \gamma$*.*
(iii) $f'(G)$ *is a strictly concave (convex) function when* $G < \gamma$ $(G > \gamma)$*; that is,* $f'''(G) \gtreqless 0$ *if and only if* $G \gtreqless \gamma$*.*
(iv) An arbitrary small neighborhood of γ*,* $(\gamma - \gamma_l,\ \gamma + \gamma_h)$ *exists, so that*

$$0 \lesssim \left| f''(G)\big|_{G \notin (\gamma - \gamma_l,\ \gamma + \gamma_h)} \right| \ll 1 \ll \Gamma, \tag{8}$$

where

$$\Gamma \equiv \left| f''(G)\big|_{G=\gamma} \right|, \tag{9}$$

$\varrho_1 \ll \varrho_2$ *means that* ϱ_1 *is much less than* ϱ_2*,* $\gamma_l,\ \gamma_h > 0$*, and* Γ *is a global maximum of* $f''(G)$*.*

Definition 1 is sufficient to assume a utility function like the one depicted as a dashed line in Figure 1 that represents the case of threshold preferences. While we are unaware of a similar definition in previous studies, the most crucial condition that characterizes threshold preferences, is property (iv) of Definition 1. Property (iv) means that the curvature of $f(G)$ is large around $G = \gamma$ (i.e., Γ is large), and $f''(G)$ has a global minimum at $G = \gamma$. If property (iv) is not satisfied, the shape of a utility function can resemble the one depicted as a solid line in Figure 1 that represents the case of standard preferences.

Properties (ii) and (iii) guarantee only that the curvature of $f(G)$ is maximized at $G = \gamma$. In other words, if the curvature is not large and property (iv) is not satisfied, preferences become normal in the sense that marginal utility $f'(G)$ uniformly diminishes at any consumption level of G, even if properties (ii) and (iii) hold. Property (i) is a standard assumption in literature, and unlike Hayashi and Ohta (2007), we do not assume a finite satiety point in utility; that is, the marginal utility $f'(G)$ diminishes to zero, only when G approaches infinity.

It is worth mentioning that the conditions of both individuals' optimization and efficient allocation, (4) and (6), remain invariant even when considering threshold preferences, because Definition 1 is within the standard framework of the preceding section and consistent with the standard assumption of $f(G)$ (i.e., the law of diminishing marginal utility holds for any G).

4 Standard versus threshold preferences

Subsequently, we examine the discrepancies between Nash and optimal provisions in a relationship with group size. In order to highlight the difference in the results between standard and threshold preferences, clear comparisons between the two cases are provided.

4.1 Constant marginal costs

We begin by examining the instance in which marginal costs are constant. In this case, the right-hand side (RHS) of (4) is θ and that of (6) is θ/n. Thus, G^* increases as n increases, whereas $\overline{G}$ does not depend on n; accordingly, an inverse relationship exists between group size and efficiency. This inverse relationship can also be confirmed in terms of group size elasticity. Since $c'' = 0$, the Nash group size elasticity in (5) becomes zero. In contrast, the optimal group size elasticity in (7) is $-\theta/\left[nG^* f''(G^*)\right]$, while it takes positive values and converges to zero as $n \to \infty$.

[Insert Figure 2 around here]

Thus, $G^* - \overline{G}$ inevitably increases as n increases due to the law of diminishing marginal utility ($f''(G) < 0$); however, given the group size n, threshold preferences could alleviate the inefficiency of the private provision of a public good. To confirm our suspicion, we depict the representative shape of (4) and (6) as Figure 2, in which Panels A and B respectively show the cases of standard and threshold preferences.[4] Under standard preferences, the marginal utility $f'(G)$ immediately decreases, and the left-hand side (LHS) of (4) and (6) has a noticeable negative slope even when G is small. In contrast, the slope of marginal utility is flatter for small values of G and steeper around $G = \gamma$ in the case of threshold preferences, compared to standard preferences. As a result, given any finite group size n, the Nash equilibrium provision $\overline{G}$ increases and approaches the optimal provision G^*.

In particular, the following proposition can be proved.

Proposition 1. *Suppose that $U(x, G)$ is the threshold utility function, marginal costs are constant, and Assumption 1 holds. For any finite n, we have*

$$\lim_{\Gamma \to \infty} (G^* - \overline{G}) \to 0. \tag{10}$$

Proof. When taking a limit $\Gamma \to \infty$, solution of $f'\left(\overline{G}\right) = \theta$ converges to γ. In the limit, given any finite n, solution of $f'(G^*) = \theta/n$ converges to γ. Therefore, $G^* - \overline{G}$ approaches zero, when $\Gamma \to \infty$. □

Proposition 1 implies that if the curvature of threshold preferences, Γ, is sufficiently large, the discrepancy of $\overline{G}$ and G^* is negligible regardless of group size n. The situation of the limit $\Gamma \to \infty$ is indicated by the dashed line in Panel B of Figure 2, and we confirm the results visually as well.

[4] θ and θ/n do not intersect in Figure 2, because $n > 1$ and accordingly $\theta/n < \theta$.

4.2 Increasing marginal costs

Referring to the case of increasing marginal costs, the slope of the RHS of (4) is that $\partial c'/\partial G = c''/n$, whereas the slope of the RHS of (6) is that $\partial (c'/n)/\partial G = c''/n^2$ and smaller than that of (4) for all G. To simplify the analysis of the case of increasing marginal costs, we assume the following condition.

Assumption 3. *If $c'(g)$ increases and $c''(g) > 0$, then*
(i) $\lim_{g\to 0} c''(g) = \xi$ where $0 < \xi < \infty$.
(ii) $c''(g) + gc'''(g) > 0$.

Condition (ii) of Assumption 3 holds except that $|c'''(g)| \geq c''(g)/g$ when $c'''(g) < 0$. As in the next lemma, Assumption 3 ensures that both slopes of the RHS of (4) and (6) decreases as n increases and converges to zero as $n \to \infty$.

Lemma 1. *Suppose that marginal costs increase and that Assumptions 2 and 3 hold. Then, we have*
(i) $\lim_{n\to\infty} \partial c'/\partial G = 0$ and $\lim_{n\to\infty} \partial (c'/n)/\partial G = 0$.
(ii) $\partial c'/\partial G$ and $\partial (c'/n)/\partial G$ decreases as n increases.

Proof. First, consider the limits when $n \to \infty$. Since $\lim_{g\to 0} c''(g)$ is finite, we obtain

$$\lim_{n\to\infty} \frac{\partial c'}{\partial G} = \lim_{n\to\infty} \frac{c''}{n} = 0, \tag{11}$$

and

$$\lim_{n\to\infty} \frac{\partial (c'/n)}{\partial G} = \lim_{n\to\infty} \frac{c''}{n^2} = 0. \tag{12}$$

Next, consider the partial derivative of the slopes with respect to n. From $c''(g)+gc'''(g) > 0$,

$$\frac{\partial^2 c'}{\partial n \partial G} = -n^{-2}\left[c'' + gc'''\right] < 0. \tag{13}$$

Moreover, since $2c''(g) + gc'''(g) > c''(g) > 0$ holds, we have

$$\frac{\partial^2 (c'/n)}{\partial n \partial G} = -n^{-3}\left[2c'' + gc'''\right] < 0. \tag{14}$$

□

Lemma 1 implies that $\overline{G}$ as well as G^* strictly increases in n, unlike the case of constant marginal costs. As a result, even when assuming increasing marginal costs, the introduction of threshold preferences could make the voluntary provision less suboptimal.

[Insert Figure 3 around here]

As before, Figure 3 shows the representative shape and illustrates how the introduction of threshold preferences alleviates suboptimality.[5] Panels A and B depict standard and threshold preferences, respectively. The figure exemplifies that if we assume threshold preferences, then $\overline{G} - G^*$ shrinks for a certain range of n.

The dashed line in Figure 3 depicts $f'(G)$ in the case of the limit $\Gamma \to \infty$. For the limit, the next proposition is obtained.

Proposition 2. *Suppose that $U(x, G)$ is the threshold utility function, marginal costs increase, and Assumptions 2 and 3 hold. For any n which is larger than n_γ where $f'(0) = c'(\gamma/n_\gamma)$ is satisfied, we have*

$$\lim_{\Gamma \to \infty} (G^* - \overline{G}) \to 0. \tag{15}$$

Proof. If $n > n_\gamma$, solution of $f'\left(\overline{G}\right) = c'(\overline{g})$ converges to γ as $\Gamma \to \infty$. Moreover, since $c'(g) > c'(g)/n$, we obtain

$$f'(0) = c'(\gamma/n_\gamma) > c'(\gamma/n_\gamma)/n_\gamma. \tag{16}$$

From this, if $n > n_\gamma$, solution of $f'(G^*) = c'(g^*)/n$ converges to γ as $\Gamma \to \infty$. Therefore, $G^* - \overline{G}$ approaches zero, when $\Gamma \to \infty$. □

Proposition 2 means that if the curvature of threshold preferences, Γ, is sufficiently large, the discrepancy of $\overline{G}$ and G^* is negligible for any n that is larger than n_γ.

We next investigate how the inefficiency $G^* - \overline{G}$ relates to the group size n. First, noting that $\lim_{n\to 1} \overline{G} = \lim_{n\to 1} G^*$, we denote the limits as G_{inf}. It is obvious that if we consider a situation so that $G_{inf} \geq \gamma$, then $G^* - \overline{G}$ monotonically increases with n. To exclude the monotonicity case, we assume the following condition:

Assumption 4. *If $c'(g)$ increases and $c''(g) > 0$, then $G_{inf} < \gamma$.*

Assumption 4 implies that the threshold level γ is not an extremely small quantity.

[Insert Figure 4 around here]

Importantly, while G^* is larger than $\overline{G}$ for any n, $G^* - \overline{G}$ is not monotonically increasing with n, provided that Γ is sufficiently large and Assumption 4 is satisfied in addition to

[5] Since $n > 1$, $c'(G/n)$ and $c'(G/n)/n$ do not intersect in Figure 3, except for the case where $c'(0) = 0$ and they intersect at $G = 0$.

Assumptions 2 and 3. The reason for non-monotonicity is straightforward, and we can understand this by considering the potential three stages depending on group size.

Suppose that Γ is sufficiently large. The first stage is the situation in which the group size is small so that $\overline{G}$ and G^* are much less than γ, and accordingly, $f''(\overline{G}) = f''(G^*) \simeq 0$, as shown in Panel A of Figure 4. In this case, since $f''(\overline{G}) = f''(G^*) \simeq 0$, we can respectively reformulate the group size elasticity in (5) and (7) as

$$\frac{d\overline{G}}{dn}\frac{n}{\overline{G}} \simeq 1, \tag{17}$$

$$\frac{dG^*}{dn}\frac{n}{G^*} \simeq 1 + \frac{nc'(g^*)}{G^* c''(g^*)}. \tag{18}$$

Since $nc'(g^*) / [G^* c''(g^*)] > 0$, it evidently follows that the optimal group size elasticity is larger than the Nash group size elasticity. Consequently, $G^* - \overline{G}$ increases with n when n is small.

In the second stage, in which the group size n is large to a certain extent and only G^* reaches the values in close proximity to the threshold level γ as shown in Panel B of Figure 4, the optimal group size elasticity in (7) becomes virtually zero because $f''(G^*)$ takes large negative values. On the contrary, $\overline{G}$ remains smaller than γ and $f''(\overline{G}) \simeq 0$, and the Nash group size elasticity in (5) remains approximately 1. Thus, the group size elasticity in (5) and (7) can be rewritten respectively as

$$\frac{d\overline{G}}{dn}\frac{n}{\overline{G}} \simeq 1, \tag{19}$$

$$\frac{dG^*}{dn}\frac{n}{G^*} \simeq 0. \tag{20}$$

In this case, the Nash group size elasticity is larger than the optimal group size elasticity, and $G^* - \overline{G}$ decreases with n. Incidentally, in a limiting case where $\Gamma \to \infty$, as shown by the dashed line in Panel B of Figure 4, the Nash equilibrium provision and the efficient provision converges as per Proposition 2.

The last stage, as shown in Panel C of Figure 4, is a situation in which the group size n is larger than in the second stage. In this case, $f''(G^*) \lesssim 0$, and $f''(\overline{G})$ takes large negative values. Moreover, it holds that $c''(g^*)/n \simeq 0$, and $nf''(G^*)$ takes finite negative values. Therefore, we approximately obtain

$$\frac{d\overline{G}}{dn}\frac{n}{\overline{G}} \simeq 0, \tag{21}$$

$$\frac{dG^*}{dn}\frac{n}{G^*} \simeq -\frac{c'(g^*)/G^*}{nf''(G^*)}. \tag{22}$$

Since $-c'(g^*) / [G^* nf''(G^*)] > 0$, this suggests that in the third stage, $G^* - \overline{G}$ increases with n.

The second and third stages indicate that a U-shaped relationship exists between $G^* - \overline{G}$ and n, which confirms the local efficient group size n^*. To summarize the preceding argument, we obtain the following proposition:

Proposition 3. *Suppose that $U(x, G)$ is the threshold utility function, marginal costs increase, and Assumptions 2–4 hold. If Γ is sufficiently large, then a unique n^* that locally minimizes $G^* - \overline{G}$ exists.*

This result is remarkable and poses important implications. A monotonically inverse relationship between efficiency and group size in the private provision of public goods is a widely accepted result; that is, a large group size aggravates the free-rider problem. However, Proposition 3 indicates that even if the group size becomes large, the efficiency could be improved. This implies, for example, that there are nations or local municipalities with suboptimal population size n^*, when certain national or local public goods are voluntarily provided.

[Insert Figures 5 and 6 around here]

It is intuitive that n^* depends on γ. Figure 5 illustrates the effect of an increase in γ. In this case, $f'(G)$ locus shifts right according to the amount of the increase in γ from γ_1 to γ_2 (i.e., $\gamma_2 > \gamma_1$). When $\gamma = \gamma_1$, $G^* - \overline{G}$ is $G_1^* - \overline{G}_1$ and small. When $\gamma = \gamma_2$, $G^* - \overline{G}$ becomes $G_2^* - \overline{G}_2$ and larger. Thus, while a rise in γ increases $G^* - \overline{G}$, it produces the situation where $G^* - \overline{G}$ could substantially decrease by increasing n. This suggests that n^* becomes larger as γ increases.

Moreover, n^* depends on the shape of the cost function. In particular, slopes of marginal costs, $c''(g)$, appear to be crucial to n^*. Figure 6 exemplifies the case of increases in the slopes of marginal costs.[6] Due to the increases, both G^* and $\overline{G}$ decrease, and there is scope to decrease $G^* - \overline{G}$ by increasing n. In other words, n^* becomes larger as the slopes of marginal costs increase. Thus, from Proposition 3, the following corollaries are immediately apparent.[7]

Corollary 1. *Suppose that $U(x, G)$ is the threshold utility function, marginal costs increase, and Assumptions 2–4 hold. Then, n^* is proportional to γ.*

Corollary 2. *Suppose that $U(x, G)$ is the threshold utility function, marginal costs increase, and Assumptions 2–4 hold. Then, n^* is proportional to $c''(g)$.*

[6]For example, if we specify that $c(g) = \frac{\phi}{2}g^2 + \eta g$, an upward shift of $c'(g)$ occurs according to an increase in the slopes $c''(g) = \phi$. In this specification, consistent with Assumption 2, it holds that $c'(0) = \eta \geq 0$ and $c''(g) = \phi > 0$.

[7]Refer to the Appendix for further analyses using numerical methods.

5 Discussion

As per the introduction, present threshold preferences seem plausible in some issues such as volunteering activities and cleaning the environment, because marginal utility can be considered large and rarely diminished below a threshold level in these issues. However, present threshold preferences are narrowly defined, without matching the situations of threshold public goods that are effectively provided only if a certain number of total contributions are made. Examples of threshold public goods include public transportation infrastructure, such as roads and airports, that cannot be provided and used before the completion of construction. Regarding this connection, for example, Spencer *et al.* (2009) mention that "it would not make much sense to provide half a lighthouse, two-thirds of a bridge, or one-quarter of a trail needed for a public bike path to connect two towns."

[Insert Figure 7 around here]

In the case of threshold public goods, the utility would rapidly increase in close proximity to the threshold level (also referred to as provision point). This situation resembles the depiction of the dashed line in Figure 7. The upper panel shows the possible shape of $f(G)$, and the lower panel shows $f'(G)$ corresponding to such a shape of $f(G)$. Note that this type of threshold preference is outside the framework considered above in that the law of diminishing marginal utility (i.e., $f''(G) < 0$) no longer holds for any G. However, we conjecture that our main results are applicable to this threshold preference, because both the RHS of (4) and that of (6) equals their LHS (i.e., $f'(G)$) in close proximity to the threshold level. Specifically, in the lower panel of Figure 7, the RHS in both cases would intersect the LHS in close proximity to the threshold level, because the LHS exhibits a positive spike in the threshold level. As a result, the discrepancies between Nash and optimal provisions would be negligible for any group size.[8]

6 Conclusion

Since Samuelson's (1954) influential article, most undergraduate public economics textbooks state that public goods are underprovided in static games with voluntary contributions and that inefficiency arises in a general context. Moreover, there is now a general consensus in existing literature that the relationship between inefficiency and group size is monotonically increasing. Although there is no doubt about the validity of such a consensus in general, this study has shed new light on this fundamental issue in particular

[8]It should be noted that such an intersect would not exist when considering the case of marginal costs that are largely increasing and a small group size.

cases. In other words, there are plausible cases in which inefficiency can be substantially lessened and in which the monotonically increasing relationship is broken. To present the results, we analyzed standard models except for a newly proposed preference of individuals, referred to as threshold preference. Although the proposed utility function seems plausible in some types of public goods and satisfies standard assumptions such as the law of diminishing marginal utility, it nevertheless alleviates inefficiency. Furthermore, if we additionally assume increasing marginal costs which also seem plausible in some cases, a local efficient group size is confirmed, in contrast to the general consensus.

In order to focus on the role of threshold preferences, we have considered a minimal model of homogeneous economy. However, heterogeneous agents are common in literature on public good provision in voluntary contributions, and have been increasingly investigated in recent years (e.g., Liu, 2018, 2019; Buchholz and Liu, 2020). Thus, analyzing the role of threshold preferences in heterogeneous economies could be subjects for future research.

Appendix

Section 4 analytically shows that threshold preferences could allow the alleviation of inefficiency in the voluntary provision of public goods. In particular, if we assume threshold preferences and increasing marginal costs, the general consensus undergoes modification. That is, the monotonically increasing relationship between inefficiency and group size no longer holds.

In this appendix, we undertake several numerical analyses of the model, focusing on the case of increasing marginal costs. Our purpose is to illustrate the qualitative effects of threshold preferences on inefficiency and to present further results that are analytically ambiguous. For example, although the analysis thus far confirmed the local efficient group size n^*, we still lack an understanding of the extent to which the local efficient level deviates from the global efficient level. First, we consider the specification of threshold preferences.

Parameterization

Up to this point, $f(G)$ is not a specific function. To study this numerically, we specify threshold preferences. Note that, as shown below, the marginal utility of the following specified utility function has the form of those graphs that seem to mirror images of the graphs of the logistic function with reference to the vertical axis. For this reason, we call this an axisymmetric-logistic utility function, which has the following form.

Definition 2. *An axisymmetric-logistic utility function is defined as*

$$U(x,G) = x - \frac{\alpha}{\beta}\ln\left(1+\exp\left[-\beta(G-\gamma)\right]\right), \tag{23}$$

with $\alpha, \beta, \gamma > 0$.

Lemma 2. *Consider the axisymmetric-logistic utility function. Then, it holds that*
(i) the marginal utility, $\partial U/\partial G$*, is a twice continuously differentiable function and satisfies the law of diminishing marginal utility.*
(ii) The marginal utility has a unique inflection point $(\gamma, \alpha/2)$.
(iii) If $G < \gamma$ $(G > \gamma)$*, the marginal utility is a strictly concave (convex) function.*

Proof. The law of diminishing marginal utility can be confirmed straightforwardly, so that

$$\frac{\partial U}{\partial G} = \frac{\alpha\exp\left[-\beta(G-\gamma)\right]}{1+\exp\left[-\beta(G-\gamma)\right]} > 0, \tag{24}$$

$$\frac{\partial^2 U}{\partial G^2} = -\frac{\alpha\beta\exp\left[-\beta(G-\gamma)\right]}{\left(1+\exp\left[-\beta(G-\gamma)\right]\right)^2} < 0. \tag{25}$$

Moreover, since we have

$$\frac{\partial^3 U}{\partial G^3} = \frac{\alpha\beta^2\exp\left[-\beta(G-\gamma)\right]\left(1-\exp\left[-\beta(G-\gamma)\right]\right)}{\left(1+\exp\left[-\beta(G-\gamma)\right]\right)^3} \gtreqless 0 \quad \Leftrightarrow \quad G \gtreqless \gamma, \tag{26}$$

it is obvious that a unique inflection point of $\partial U/\partial G$ is $(\gamma, \alpha/2)$. □

[Insert Figure 8 around here]

Lemma 2 means that this felicity function has the properties stated in Definition 1 of threshold preferences. Key parameters that characterize threshold preferences, are β and γ. The curvature of the marginal utility is determined by β, and γ represents the threshold levels. To confirm these, Figure 8 plots (23) and (24) when $\alpha = 1$ and $\gamma = 5$ in conjunction with some values of β. When $\beta = 1$, the curvature of the utility function appears relatively mild, and the marginal utility declines across the board values of G. If we examine the case of $\beta = 2$, the marginal utility declines at a narrower range centering around $G = \gamma = 5$. When $\beta = 10$, the utility function is almost linear except in the vicinity of $G = \gamma = 5$, and accordingly, the marginal utility exhibits a sharp decline near the threshold level. Thus, we have the next lemma.

Lemma 3. *Consider the axisymmetric-logistic utility function. Then, when* β *is large, this function is regarded as a form of threshold utility function.*

To present quantitative analysis, specification of $c(g)$ is also required. We then specify the cost for the voluntary contribution to be quadratic

$$c(g) = \frac{\phi}{2}g^2 + \eta g, \tag{27}$$

where $c'(0) = \eta \geq 0$ and $c''(g) = \phi > 0$.

From (4) and (6), the voluntary and optimal provision are respectively obtained by solving the following equations

$$\frac{\alpha \exp\left[-\beta(\overline{G} - \gamma)\right]}{1 + \exp\left[-\beta(\overline{G} - \gamma)\right]} = \phi\overline{g} + \eta, \tag{28}$$

$$\frac{\alpha \exp\left[-\beta(G^* - \gamma)\right]}{1 + \exp\left[-\beta(G^* - \gamma)\right]} = \frac{\phi g^* + \eta}{n}. \tag{29}$$

Since these cannot be solved analytically, we present the results as numerical solutions below.

There are five parameters that characterize the equilibria. Henceforth, unless otherwise noted, we set $\alpha = 1$, $\beta = 1$, and $\gamma = 100$ for threshold preferences; similarly, $\phi = 1$ and $\eta = 0$ for the cost function.

Effects of threshold preferences

The first step includes an illustration of the basic results that suboptimality is improved when individuals have threshold preferences. Figure 9 shows the relationship between group size n and inefficiency $G^* - \overline{G}$ for various parameters of β. Note that the figure is a double logarithmic plot, and the minimum of n is set to 2. Consistent with Proposition 2, $G^* - \overline{G}$ shrinks considerably when β is large, overall. When n is small and less than approximately 10, $G^* - \overline{G}$ rises. However, when β is large, these rises appear much less pronounced. Furthermore, consistent with Proposition 3, when $\beta = 10^{-1}$, 10^0, and 10^1, a unique n^* (local efficient group size) exists between 10^2 and 10^3. More importantly, in addition to the illustrations of Propositions 2 and 3, we find that when $\beta = 10^1$, n^* is not a local minimum but a global minimum.

[Insert Figure 9 around here]

According to Corollary 1, higher threshold levels cause higher local efficient group sizes. Figure 10 illustrates this point by plotting the relationship between n and $G^* - \overline{G}$ for various γ. By definition, the degree of n^* becomes higher as γ becomes higher. A striking pattern that emerges from the figure is that $G^* - \overline{G}$ undergoes a much larger change for higher γ; consequently, in contrast to the widespread consensus, a large group size could substantially improve the efficiency of the private provision of public goods.

[Insert Figure 10 around here]

Similarly, according to Corollary 2, steeper slopes of marginal costs cause higher local efficient group sizes. In fact, Figure 11 shows that the degree of n^* becomes higher as ϕ increases, illustrating the result in Corollary 2. Compared to changes in γ (Figure 10), maximum values of $G^* - \overline{G}$ are less influenced by changes in ϕ.

[Insert Figure 11 around here]

Finally, Figure 12 explores their relationships for various η. As expected, we notice qualitatively similar results to the case of ϕ, but the effects of η appear quantitatively negligible. In comparison to Hayashi and Ohta (2007), what is more noteworthy is that the outcomes are less sensitive to $c'(0)$ (i.e., η in the present case). In the framework of Hayashi and Ohta (2007), the assumption that $c'(0) = 0$ is indispensable for achieving a notable conclusion that the inefficiency converges to zero as the group size approaches infinity. In contrast, our results are robust to the situation in which $c'(0) > 0$.

[Insert Figure 12 around here]

Overall, the numerical results consistently exhibit that if individuals have threshold preferences, the increasing relationship between group size and inefficiency is different depending on the group size; comparing the increasing phases, aggravation of efficiency is more severe when the group size is small as opposed to large. This offers a new insight that the free-rider problem becomes less serious when the group size is large rather than small.

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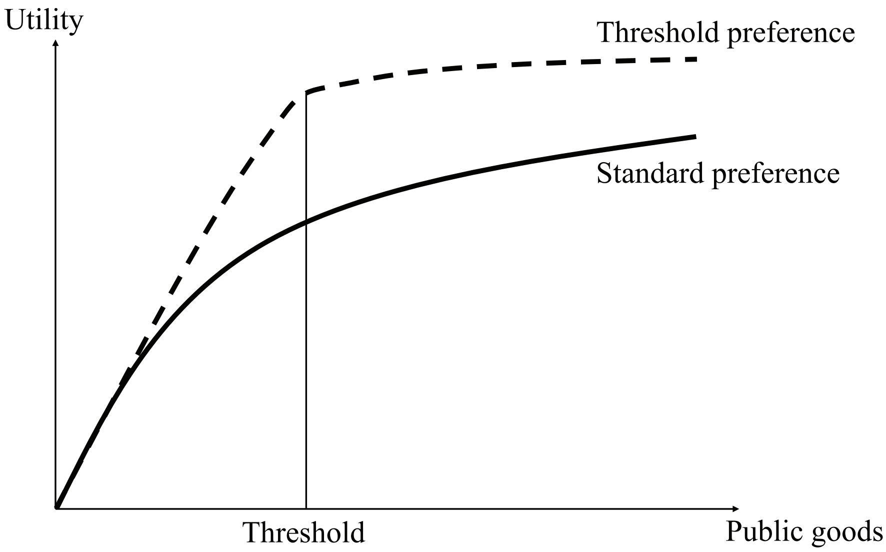
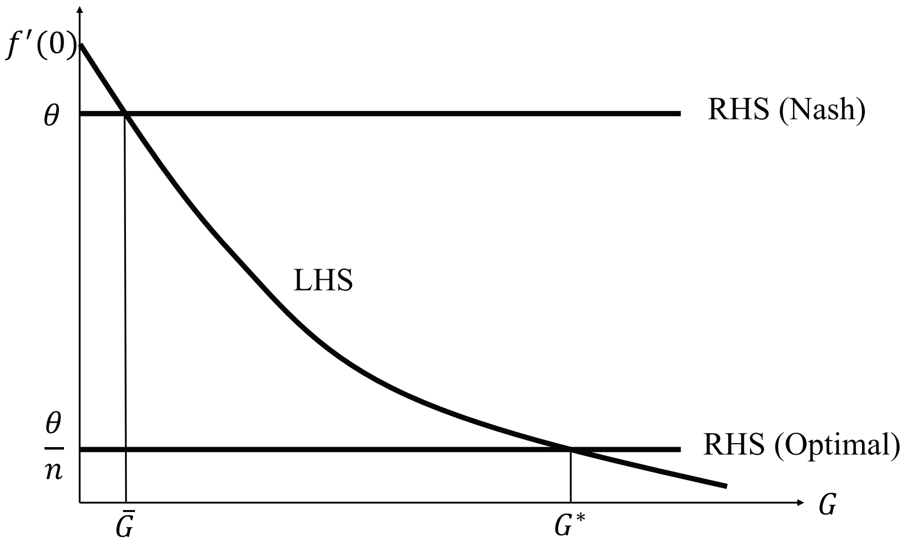


Figure 1: Threshold preference

A



B

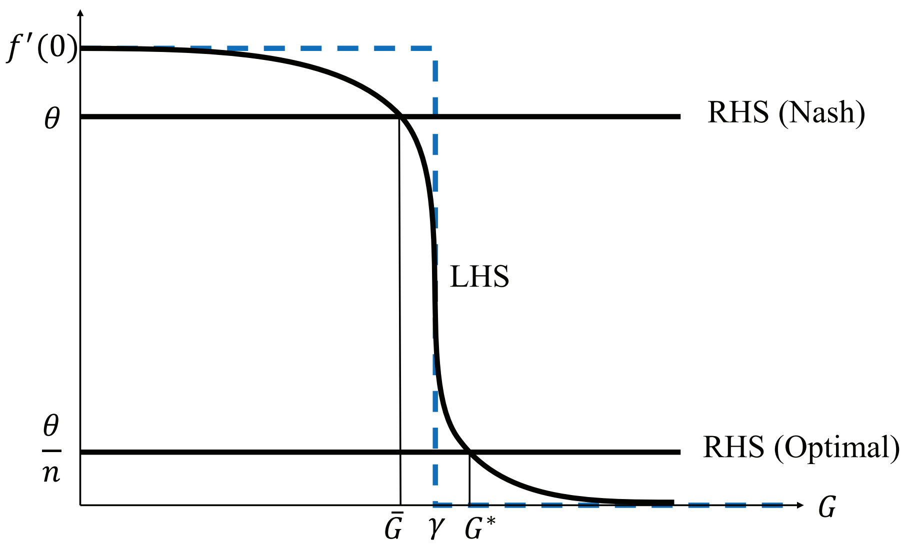
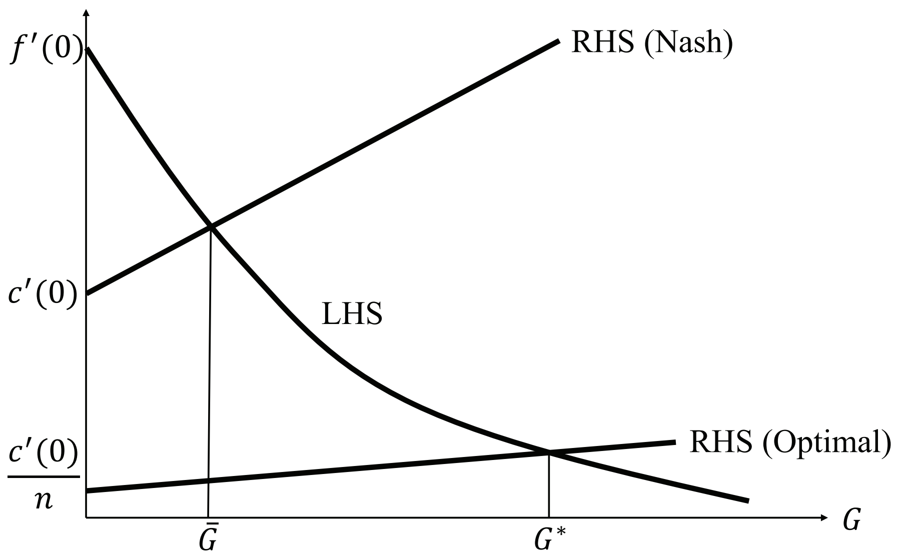


Figure 2: Case of constant marginal costs

A



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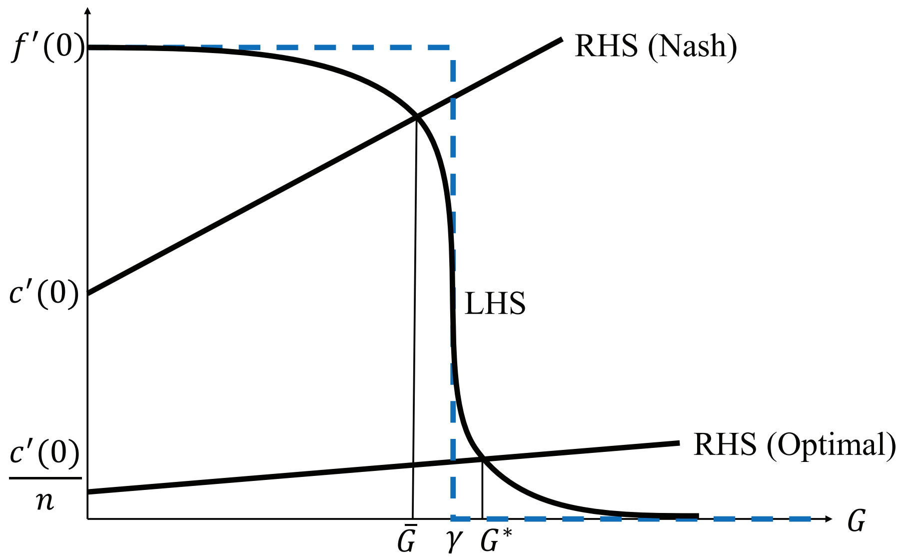

Figure 3: Case of increasing marginal costs

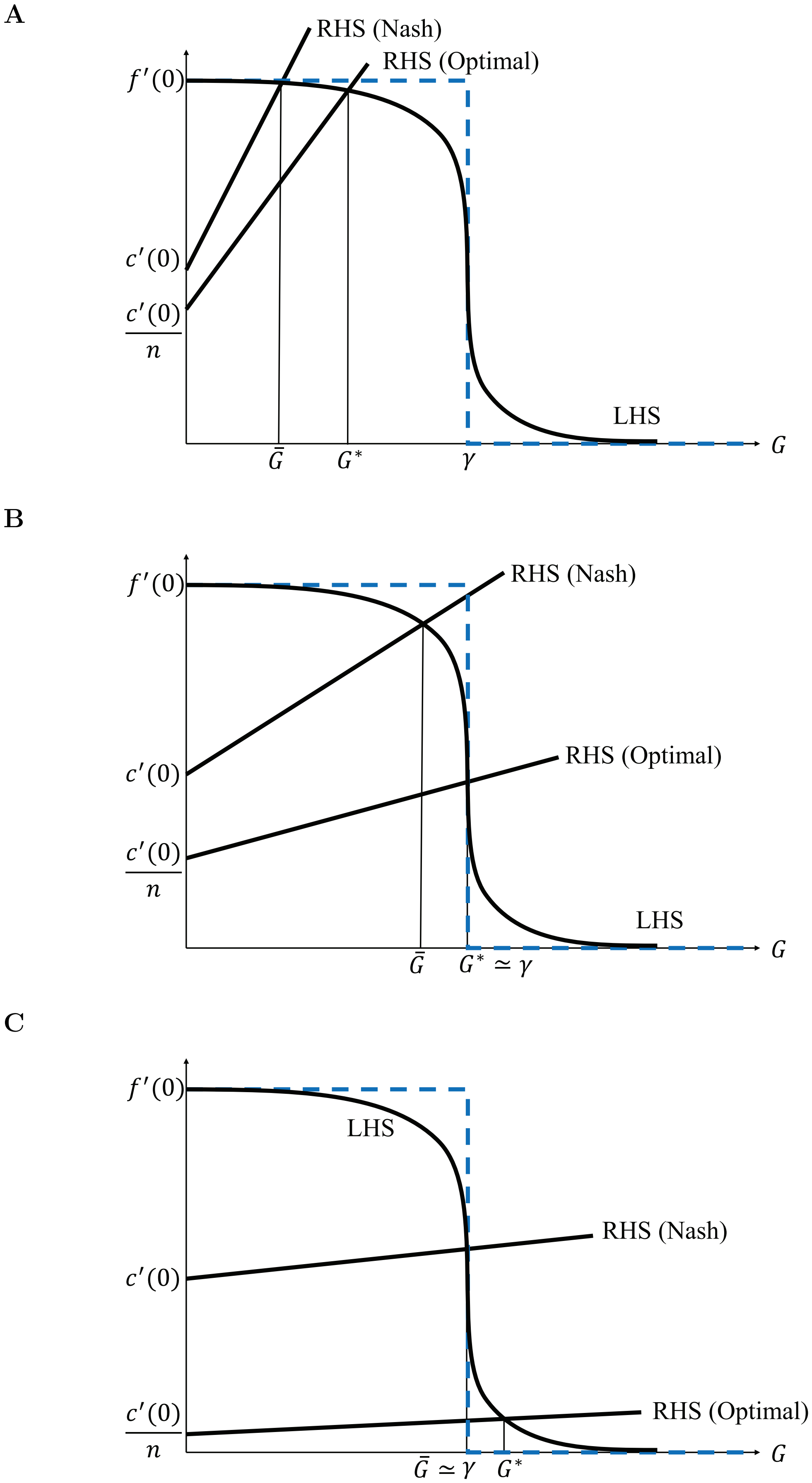

Figure 4: Three phases in case of increasing marginal costs

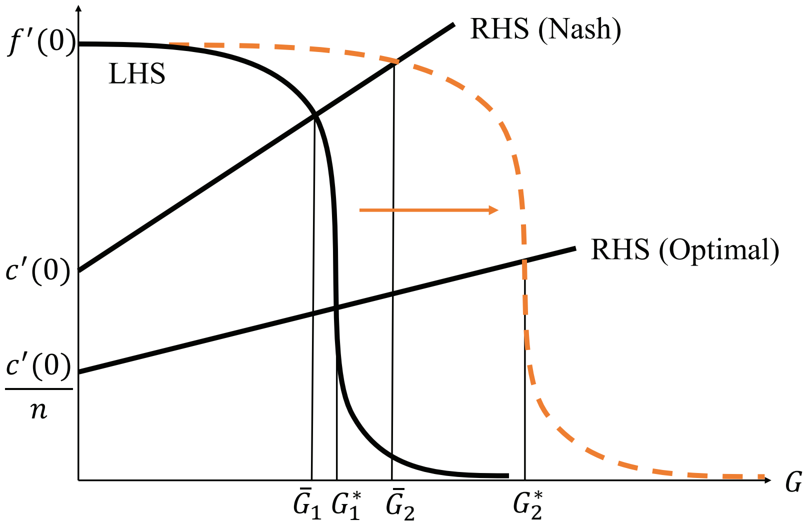

Figure 5: The effects of increases in threshold levels

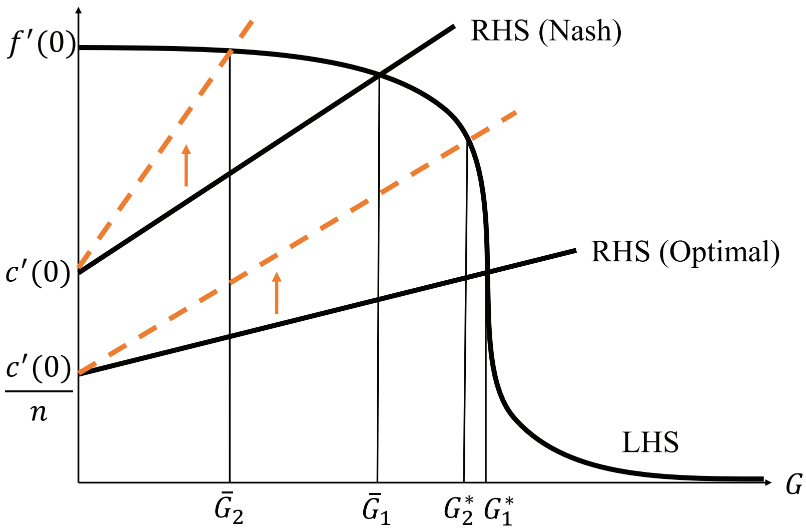

Figure 6: The effects of increases in slopes of marginal costs

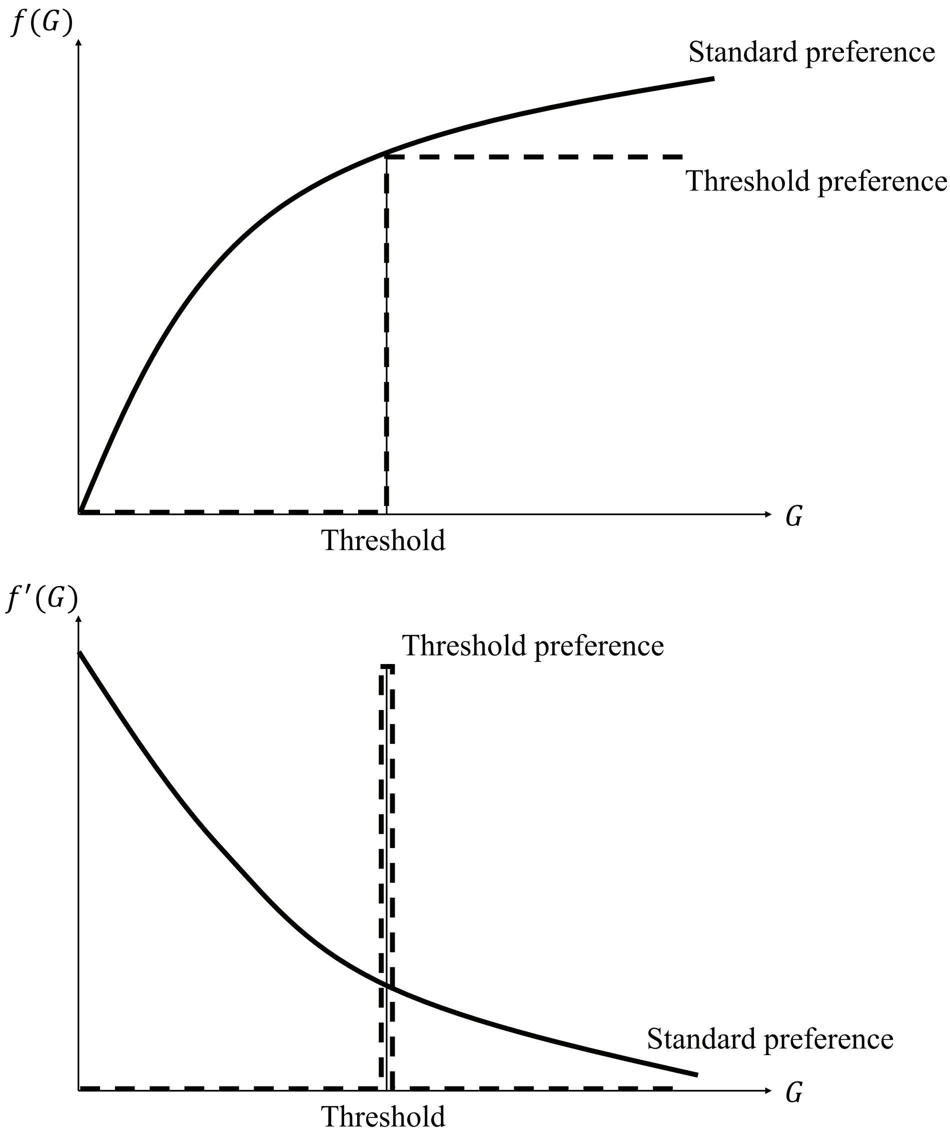

Figure 7: Other threshold preferences

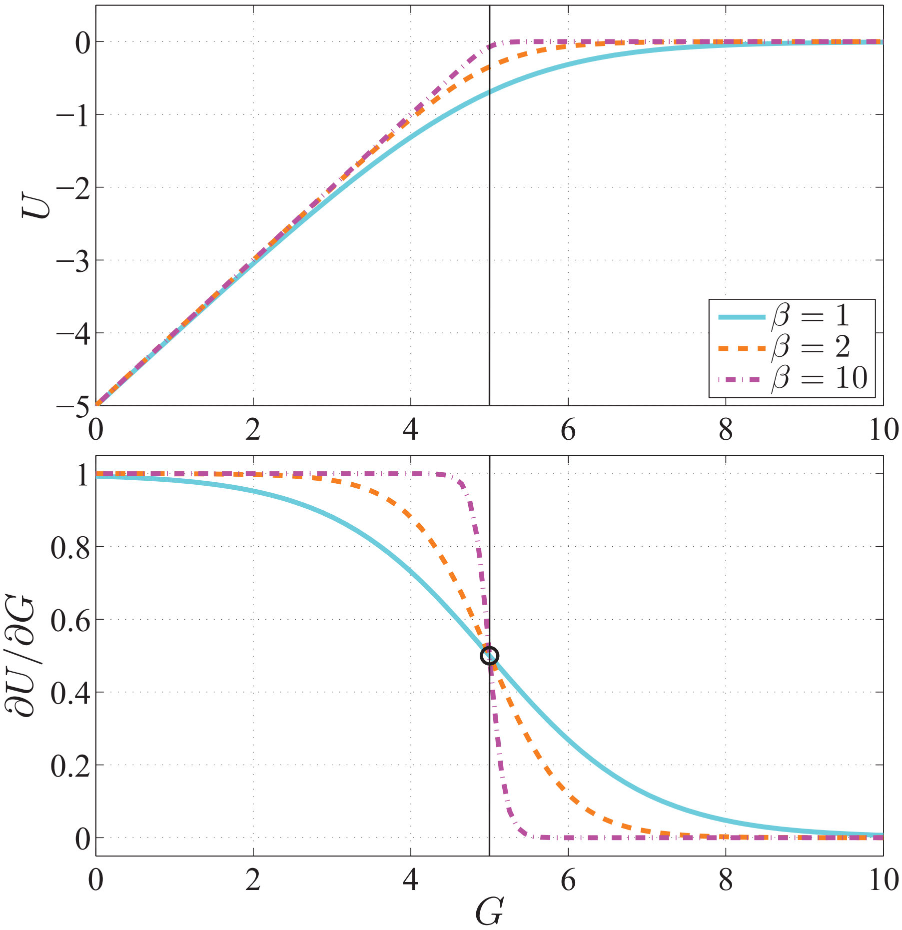

Figure 8: Axisymmetric-logistic utility function

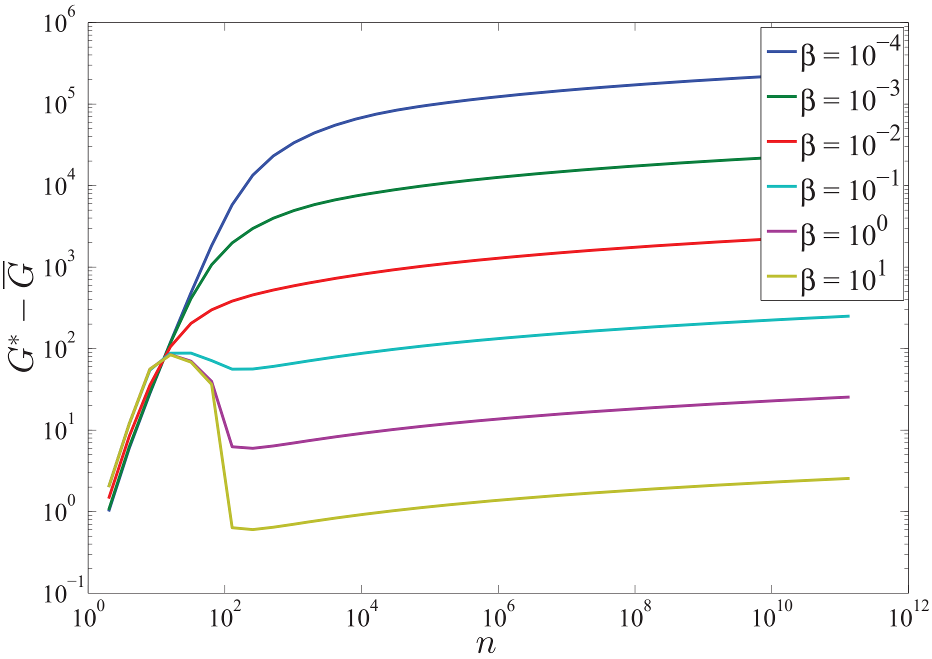

Figure 9: Group size and suboptimality for various β

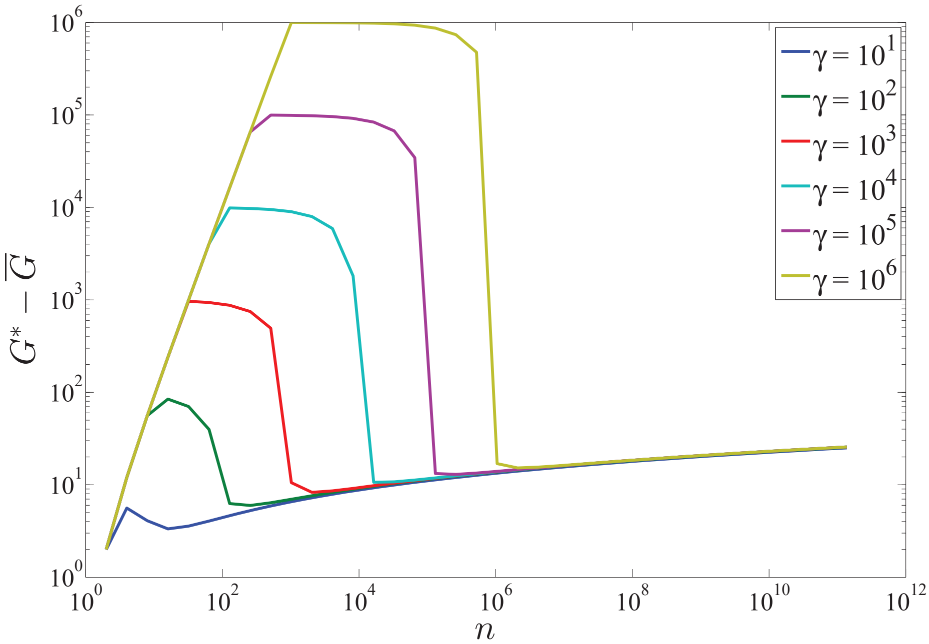

Figure 10: Group size and suboptimality for various γ

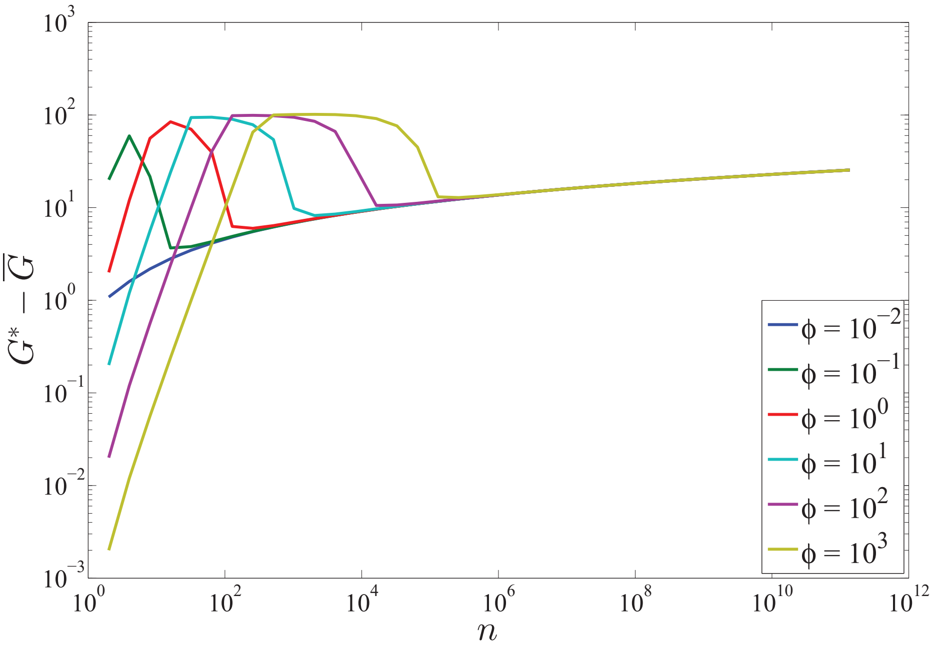

Figure 11: Group size and suboptimality for various ϕ

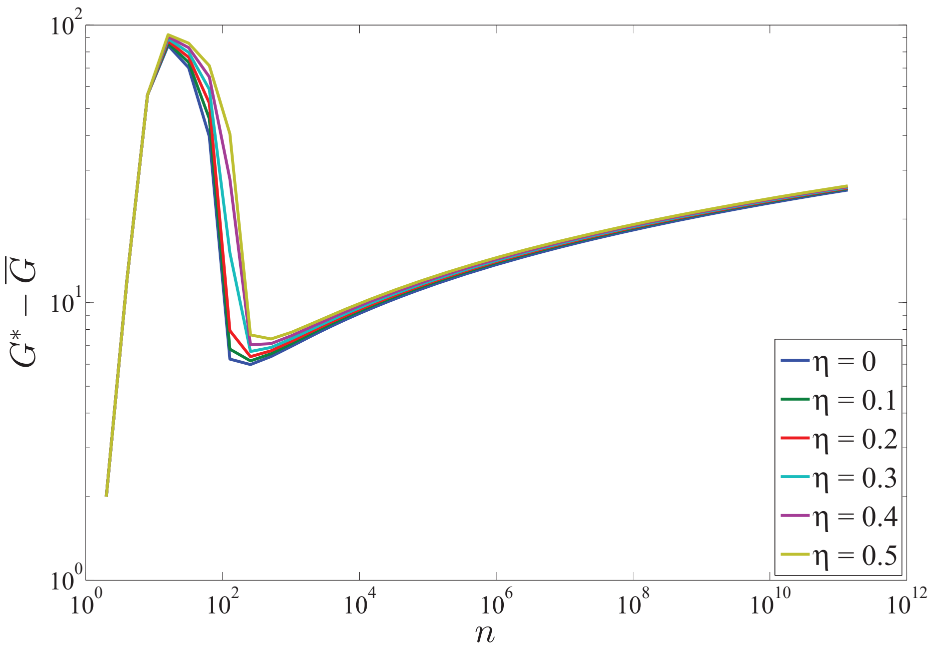

Figure 12: Group size and suboptimality for various η