Properties of the Equilibrium Revenues in Buy Price Auctions

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Abstract

This paper considers second-price, sealed-bid auctions with a buy price. We use a two-bidder framework in which bidders’ types are uniformly distributed. In general, there are many equilibria. We then look at a certain kind of pure strategy equilibrium including a truth-telling strategy equilibrium, and characterize the set of the equilibrium revenues. It is well-known that in second-price, sealed-bid auctions, the minimum equilibrium revenue is zero. We show that by introducing any effective buy price, a seller can always obtain a positive equilibrium revenue. This result gives a new reason why sellers set a buy price so often in Internet auctions.

JEL classification: D44

Key words: Auction, buy price

1 Introduction

Budish and Takeyama (2001) first consider second-price, sealed-bid auctions with a buy price. They use a simple two-bidder, two-type framework, and show that a seller who faces risk-averse bidders can improve her expected revenue by properly introducing a buy price. After Budish and Takeyama (2001), Hidvég, Wang and Whinston (2006) and Reynolds and Wooders (2009) examine English auctions with a buy price in a more general framework, where there are n bidders whose types are continuously distributed. They also show that bidders’ risk-aversion plays a key role in improvement on a seller’s expected revenue.

In Internet auctions, however, it would be difficult for a seller to introduce a buy price properly. This is because she cannot know the probability distribution of bidders’ types

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appropriately. Also, even if she can know the type distribution, as Budish and Takeyama (2001) and others show, she cannot improve her expected revenue in the case that bidders are risk-neutral. These facts suggest that in practice a seller may not be able to improve her expected revenue by introducing a buy price. Nevertheless, we often find that buy prices are set in Internet auctions.

We then attempt to examine effects of a buy price from another point of view. For this purpose, we focus on multiple equilibria. We often limit our attention to a weakly dominant strategy equilibrium in auction theory. This is not because it is enough to consider a certain equilibrium, but because multiplicity of equilibria makes an analysis complicated. Budish and Takeyama (2001) analyse a certain symmetric equilibrium such that bidders whose types are not less than the buy price bid it and bidders whose type are less than the buy price bid their own valuations. Though this symmetric equilibrium is reasonable, there are indeed many kinds of other equilibria. To examine an equilibrium as much as possible, we allow that strategies are not differentiable. Specifically speaking, we take into account strategies such that all types in some interval bid the same bid. In the analysis, we look at many pure strategy equilibria including the truth-telling strategy equilibrium, and characterize the set of the equilibrium revenues.

Milgrom (1981) points out that in second-price, sealed-bid auctions without a buy price, there are many equilibria including the truth-telling strategy equilibrium, giving simple examples in a two-bidder framework where bidders’ types are continuously distributed. We pay attention to this fact, and then contrast between the cases of second-price, sealed-bid auctions with and without a buy price to highlight the effects of a buy price.

In second-price, sealed-bid auctions without a buy price, we show that the set of the equilibrium revenues is a closed interval which contains zero. This result is obtained by constructing an equilibrium in which a seller’s expected revenue corresponds to each element in that interval. Moreover, we show that the maximum of the set of the equilibrium revenues is equivalent to the one at the truth-telling strategy equilibrium. By introducing a reserve price, off course, a seller can make the minimum equilibrium revenue positive. Even in this case, the set of the equilibrium revenues is a closed interval.

Similarly, for each buy price auction, we characterize the set of the equilibrium revenues. We show that each set of the equilibrium revenues is a closed interval in the same way as second-price, sealed-bid auctions without a buy price. We then show that a seller can always obtain a positive equilibrium revenue by setting any effective buy price. This result is obtained

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1In a common value environment, for example, Bikhchandani and Riley (1991) consider not only a symmetric equilibrium but also asymmetric equilibria. They compare a seller’s expected revenue in second-price, sealed-bid auctions with seller’s expected revenues in English auctions. Similarly, Lizzier and Persico (2000) examine various kinds of auction formats except second-price, sealed-bid auctions in an interdependent value environment. They derive necessary assumptions to obtain uniqueness results.

2Maskin and Riley (1985) show that even in a two-bidder, two-type framework, there are many equilibria.

3It is well-known that in second-price, sealed-bid auctions, there exists an equilibrium in which a seller’s expected revenue is zero.
when both the seller and bidders are risk-neutral. Therefore, we provide a new reason instead of expected revenue improvement why buy prices are so often introduced in Internet auctions.

There are several research focusing on all equilibria. Especially, there is a striking result in second-price, sealed-bid auctions where there are at least three bidders. Blume and Heidhues (2004) show that by properly introducing a reserve price, a seller can make an equilibrium unique. Obviously, this follows that an equilibrium revenue is uniquely determined. As Blume and Heidhues (2001) show, however, such kind of uniqueness result cannot be obtained in case of two bidders. Plum (1992) characterizes all equilibria in two-bidder auctions with several payment rules, which the winner pays a convex combination of the highest bid and the second highest bid. He shows that in all but second-price, sealed-bid auctions, any equilibrium is composed of continuously differentiable and strictly monotone increasing strategies. We emphasize that he does not discuss an equilibrium revenue.

According to our knowledge, there are few research considering multiplicity of equilibria in buy price auctions. Hidvégi et al. (2006) and Reynolds and Wooders (2009) restrict their attention to a certain symmetric equilibrium to consider English auctions with a buy price in which bidders’ types are continuously distributed. Inami (2011) generally examines second-price, sealed-bid auctions with a buy price in which bidders’ types are discretely distributed. In the analysis, he takes account of asymmetric equilibria. He, however, limits his attention to an equilibrium in which bidders whose types are less than the buy price bid their own valuations.

The rest of this paper is organized as follows. Section 2 describes the model. In Section 3, we show the results of second-price, sealed-bid auctions without a buy price. Section 4 examines second-price, sealed-bid auctions with a buy price. And Section 5 provides the main theorem.

2 The model

We consider second-price, sealed-bid auctions with a buy price in a basic framework. One item is auctioned by a seller. Two bidders participate in the buy price auctions. Each bidder’s type is independently and uniformly distributed. And each bidder evaluates the auctioned item, depending on their own type only.

As a buy price changes the rule of second-price, sealed-bid auctions, here we explain the rule of second-price, sealed-bid auctions with a buy price in detail. In the buy price auctions, a seller sets a buy price before the auction starts. After bidders observe the buy price, they submit a bid. If no one bids the buy price, then the highest bidder obtains the item and pays the second highest bid to the seller. If only one bidder bids the buy price, then he certainly wins the auction but must pay the buy price to the seller. If some bidders bid the same amount (it may be the buy price), we adopt a tie-breaking rule that the winner is determined with an equal probability.
Let $N = \{1, 2\}$ be the set of bidders. For each bidder $i \in N$, the set of types is $T_i = [0, \omega]$. As a seller sets a buy price $B \in [0, +\infty)$, for each bidder $i$, the set of actions is $A_i = [0, b] \cup \{B\}$. Note that the bid $\overline{b}$ is the highest bid among bids except the buy price $B$, and that we evaluate $\overline{b}$ at $B$ as an amount. In Internet auctions, bidders do not bid above the buy prices. We then assume that bidders cannot submit a bid above the buy price $B$.

For each bidder $i$, a payoff function is $u_i : A \times T_i \to \mathbb{R}$, where $A = A_i \times A_j (i \neq j)$. For each bidder $i$, a payoff is determined by

$$u_i(a; t_i) = \begin{cases} t_i - a_j & \text{if } a_i \neq B \text{ and } a_i > a_j, \\ t_i - B & \text{if } a_i = B \text{ and } a_i > a_j, \\ \frac{1}{2}(t_i - a_i) & \text{if } a_i = a_j \text{ and,} \\ 0 & \text{if } a_i < a_j, \end{cases}$$

given $t_i \in T_i$ and $a \in A$.

For each bidder $i$, a strategy is $\sigma_i : T_i \to A_i$. In the analysis, we limit our attention to pure and non-decreasing strategies. We adopt as a solution concept Bayesian Nash equilibrium. The strategy profile $\sigma = (\sigma_i(\cdot), \sigma_j(\cdot))$ is a Bayesian Nash equilibrium if for all $i \in N$, all $t_i \in T_i$, and all $a_i' \in A_i$,

$$E[u_i(a; t_i)|\sigma_i(\cdot), \sigma_j(\cdot), \rho_i(\cdot)] \geq E[u_i(a_i', a_j; t_i)|\sigma_j(\cdot), \rho_i(\cdot)],$$

where $\sigma_j(\cdot)$ is the other bidder $j$’s strategy and $\rho_i(\cdot)$ is the belief of bidder $i$’s $t_i$-type about the other bidder $j$’s type.

Blume and Heidhues (2001) show that when we consider second-price, sealed-bid auctions without a buy price $B$, there are many equilibria composed of various strategies in case of two bidders. For example, a strategy profile $\hat{\sigma} = (\hat{\sigma}_i(\cdot), \hat{\sigma}_j(\cdot))$ such that

$$\hat{\sigma}_i(t_i) = \begin{cases} t_i & \text{if } 0 \leq t_i \leq t, \\ t' & \text{if } t < t_i \leq t'', \\ t_i & \text{if } t'' < t_i \leq t''', \\ t''' & \text{if } t''' < t_i \leq \omega \end{cases}$$
and

\[\hat{\sigma}_j(t_j) = \begin{cases} 
t_j & \text{if } 0 \leq t_j \leq t, 
t & \text{if } t < t_j \leq t', 
t'' & \text{if } t' < t_j \leq t'', 
t_j & \text{if } t'' < t_j \leq t''', 
t''' & \text{if } t''' < t_j \leq \omega
t_j & \text{if } t_j < t \leq t_j. 
\end{cases}\]

is an equilibrium. Note that \(t < t' < t'' < t''' < t''''\).

In practice, it is quite difficult to analyse all pure strategy profiles. We, however, try to focus on not only a truth-telling strategy but also other strategies as many as possible. We allow that strategies are not differentiable at some points with respect to bidders’ types. Here we introduce the notion that plays an important role in the analysis.

**Definition 1.** A strategy \(\sigma_i(\cdot)\) has a plateau if there exists a type interval \([t, \tilde{t}]\) in which for each type \(t_i \in [t, \tilde{t}]\), bidder \(i\) submits the same bid.

**Assumption 1.** A strategy \(\sigma_i(\cdot)\) has at most one plateau.

As a candidate of an equilibrium, therefore, we analyse every strategy profile from ones in which both bidders play strategies that have no plateau, in other words, differentiable strategies, to ones in which both bidders play strategies that have one plateau.

3 Second-price auctions without a buy price

Our main purpose is to give a new reason why sellers set a buy price so often in Internet auctions. We need to reveal what kinds of advantage a buy price has. To highlight effects of a buy price, we contrast between the cases of second-price, sealed-bid auctions with and without a buy price. Specifically speaking, we derive the set of the equilibrium revenues in each case, and then make a detailed comparison.

At first we consider second-price, sealed-bid auctions without a buy price.

**Proposition 1.** Consider second-price, sealed-bid auctions without a buy price \(B\). Then, the set of the equilibrium revenues is \([0, \omega/3]\).

**Proof.** See Appendix.

To obtain the set of the equilibrium revenues, we have constructed an equilibrium in practice.\(^8\)

The maximum of the set is the same as an equilibrium revenue that a seller obtains at the truth-telling strategy equilibrium, and the minimum of it is 0.\(^9\) Moreover, the set of the

\(^8\)Especially, we have paid attention to a certain kind of equilibria and slightly modified it.

\(^9\)It is well-known in the literature that we can construct an equilibrium that yields zero expected revenue.
equilibrium revenues is dense. As a seller cannot know which equilibrium emerges, she risks the possibility that an equilibrium revenue is zero.

4 Second-price auctions with a buy price

In this section, we consider second-price, sealed-bid auctions with a buy price $B$. Especially, we focus on second-price, sealed-bid auctions with a buy price $B \in (0, \omega]$. In Appendix, we analyse the rest of buy price auctions—a second-price, sealed-bid auction with the buy price $B = 0$ and second-price, sealed-bid auctions with a buy price $B \in (\omega, +\infty)$, respectively. As well as the cases of second-price, sealed-bid auctions without a buy price $B$, we look at every strategy profile from ones in which both bidders play strategies that have no plateau to ones in which both bidders play strategies that have one plateau. We then derive the set of the equilibrium revenues.\(^{10}\)

In a similar way to those of second-price, sealed-bid auctions without a buy price $B$, we pick up one equilibrium to derive the set of the equilibrium revenues. Indeed, one of the most basic strategy profiles that would be expected to be an equilibrium is not an equilibrium.

**Claim 1.** Consider second-price, sealed-bid auctions with a buy price $B \in (0, \omega]$. Then, any strategy profile such that both bidders’ all types which are not less than the buy price $B$ submit it is not a Bayesian Nash equilibrium.

**Proof.** See Appendix.

By Claim 1, we need to focus on other strategy profiles, where bidders’ types that are not less than the buy price $B$ do not always submit it.

4.1 Second-price auctions with a buy price $B \in (0, \frac{\omega}{2})$

When we derive the set of the equilibrium revenues in second-price, sealed-bid auctions with a buy price $B \in (0, \omega]$, we mainly look at two different strategy profiles. Depending on the strategy profile to be analysed, we divide second-price, sealed-bid auctions with a buy price $B \in (0, \omega]$ into two classes. One is second-price, sealed-bid auctions with a buy price $B \in (0, \frac{\omega}{2})$. The other is second-price, sealed-bid auctions with a buy price $B \in [\frac{\omega}{2}, \omega]$.

First we consider second-price, sealed-bid auctions with a buy price $B \in (0, \frac{\omega}{2})$. In these

\(^{10}\)For a second-price, sealed-bid auction with the buy price $B = 0$, the set of the equilibrium revenues is $\{B\}$. For second-price, sealed-bid auctions with a buy price $B \in (\omega, +\infty)$, the set of the equilibrium revenues is equivalent to that of second-price, sealed-bid auctions without a buy price $B$. See Appendix for details.
buy price auctions, we pay attention to the strategy profile \( \sigma^t = (\sigma^t_i(\cdot), \sigma^t_j(\cdot)) \) such that

\[
\sigma^t_i(t_i) = \begin{cases} 
0 & \text{if } 0 \leq t_i \leq b, \\
t_i & \text{if } b < t_i \leq \tilde{b}, \\
\tilde{b} & \text{if } B \leq t_i \leq t^b_i, \\
B & \text{if } t^b_i < t_i \leq \omega
\end{cases}
\]

and

\[
\sigma^t_j(t_j) = \begin{cases} 
b & \text{if } 0 \leq t_j \leq b, \\
t_j & \text{if } b < t_j \leq \tilde{b}, \\
\tilde{b} & \text{if } B \leq t_j \leq t^b_j, \\
B & \text{if } t^b_j < t_j \leq \omega.
\end{cases}
\]

Here let \( t^b_i \equiv (\omega B - b^2) / (\omega - B) \) and \( t^b_j \equiv (\omega B + b^2) / (\omega - B) \), respectively. Note that \( b \in [0, \tilde{b}] \).

When bidders play the strategies \( \sigma^t_i(\cdot) \) and \( \sigma^t_j(\cdot) \), bidders’ types that are not less than the buy price \( B \) but not greater than the thresholds do not submit the buy price \( B \).

At the equilibrium \( \sigma^t \), which is shown in Appendix, the seller obtains the expected revenue

\[
E[R^B_I(b)] \equiv \frac{1}{6\omega^2} \times \left\{-2b^3 + 3(3t^b_i + t^b_j - 4B)b^2 - 6(t^b_i B + t^b_j B - 2B^2)b + 2B^3 - 3t^b_i B^2 - 3t^b_j B^2 + 6\omega^2 B\right\}.
\]

Once the bid \( b \) is fixed, then the corresponding equilibrium \( \sigma^t \) is determined. This implies that the expected revenue \( E[R^B_I(b)] \) is a function of the bid \( b \).

**Proposition 2.** Consider second-price, sealed-bid auctions with a buy price \( B \in (0, \tilde{b}) \).

Then, the set of the equilibrium revenues is

\[
\begin{align*}
[E[R^B_I(\tilde{b})], E[R^B_I(0)]] & \quad \text{if } B \in (0, (\sqrt{2} - 1)\omega], \\
[E[R^B_I(\tilde{b})], E[R^B_I(0)]] & \quad \text{if } B \in ((\sqrt{2} - 1)\omega, \tilde{b}).
\end{align*}
\]

Here let \( \hat{b} \equiv \sqrt{\omega^2 - 2\omega B} \).

**Proof.** See Appendix.

In second-price, sealed-bid auctions with a buy price \( B \in (0, (\sqrt{2} - 1)\omega] \), both \( t^b_i \in (B, \omega) \) and \( t^b_j \in (B, \omega) \) are well-defined. In second-price, sealed-bid auctions with a buy price \( B \in ((\sqrt{2} - 1)\omega, \tilde{b}) \), \( t^b_i \in (B, \omega) \) is still well-defined. However, there is a case in which \( t^b_j \in (B, \omega) \) is not well-defined. Therefore, we slightly modify the interval of the bid \( b \), considering the expected revenue \( E[R^B_I(b)] \).
4.2 Second-price auctions with a buy price $B \in \left[ \frac{\omega}{2}, \omega \right]$

Now, we consider second-price, sealed-bid auctions with a buy price $B \in \left[ \frac{\omega}{2}, \omega \right]$. In these auctions, the strategy profile $\sigma^*_{I}$ is not an equilibrium. Instead, we pay attention to the strategy profile $\sigma^b = (\sigma^{b}_{I}(\cdot), \sigma^{b}_{J}(\cdot))$ such that

$$
\sigma^{b}_{I}(t) = \begin{cases} 
0 & \text{if } 0 \leq t \leq b, \\
t & \text{if } b < t \leq \tilde{b}, \\
\tilde{b} & \text{if } B \leq t \leq \omega,
\end{cases}
$$

and

$$
\sigma^{b}_{J}(t) = \begin{cases} 
b & \text{if } 0 \leq t \leq b, \\
t & \text{if } b < t \leq \tilde{b}, \\
\tilde{b} & \text{if } B \leq t \leq \omega.
\end{cases}
$$

Note that $b \in [0, \tilde{b}]$. When bidders play the strategies $\sigma^{b}_{I}(\cdot)$ and $\sigma^{b}_{J}(\cdot)$, bidders’ types that are not less than the buy price $B$ do not submit the buy price $B$ at all.

At the equilibrium $\sigma^b$, the seller obtains the expected revenue

$$
E[R^b_{II}(b)] = \frac{1}{3\omega^2} \left(-B^3 + 3\omega^2B - 3\omega B^2 + B^3\right).
$$

As well as the case of the equilibrium $\sigma^*$, the expected revenue $E[R^B_{II}(b)]$ is a function of the bid $b$.

**Proposition 3.** Consider second-price, sealed-bid auctions with a buy price $B \in \left[ \frac{\omega}{2}, \omega \right]$. Then, the set of the equilibrium revenues is

$$
[E[R^b_{II}(\tilde{b})], E[R^B_{II}(0)]] \quad \text{if } B \in \left[ \frac{\omega}{2}, \omega \right].
$$

Here let $\tilde{b} \equiv \sqrt{-\omega^2 + 2\omega B}$.

**Proof.** See Appendix.

In second-price, sealed-bid auctions with a buy price $B \in \left[ \frac{\omega}{2}, \omega \right]$, we need to restrict the interval of the bid $b$, which guarantees the existence of an equilibrium.

5 Main theorem and discussion

We have derived for each buy price auction the corresponding set of the equilibrium revenues. In this section, we show that introducing a buy price gives sellers a distinct advantage. For
this purpose, we focus on the worst case for sellers. In other words, we consider the minimum of the set of the equilibrium revenues.\footnote{In the Appendix, we discuss the maximum of the set of the equilibrium revenues.}

**Theorem 1.** Consider second-price, sealed-bid auctions with a buy price $B \in [0, +\infty)$. Then, a seller always obtains a positive equilibrium revenue by introducing an effective buy price.

**Proof.** See Appendix.

In the case that a seller sets the buy price $B = 0$, she obtains zero equilibrium revenue. If the seller sets a buy price $B$ that is equal to greater than the highest possible valuation $\omega$, no bidder intends to submit the buy price at any equilibrium. Therefore, rational sellers will not set such buy prices. By introducing an appropriate buy price $B \in (0, \omega)$, sellers can avoid a risk that they obtain zero equilibrium revenue. We emphasize that Theorem 1 holds under the situation in which bidders are risk-neutral.

Here we focus on the minimum of the set of the equilibrium revenues of each second-price, sealed-bid auction with a buy price $B$. We can regard it as a function of the buy price $B$. Let us denote by $E[R^w(B)]$ the minimum of the set of the equilibrium revenues in a second-price, sealed-bid auction with the buy price $B$. We give an illustrative example of the case in which bidders’ types are distributed on the interval $[0, 1]$.

**Example 1.** Suppose that bidders’ types are uniformly distributed on the interval $[0, 1]$. Then, the minimum of the set of the equilibrium revenues is

$$E[R^w(B)] = \begin{cases} 
E[R^B_I(b)] & \text{if } 0 < B \leq \sqrt{2} - 1, \\
E[R^B_I(\tilde{b})] & \text{if } \sqrt{2} - 1 < B \leq \frac{1}{2}, \\
E[R^B_{II}(\tilde{b})] & \text{if } \frac{1}{2} < B \leq 1, \\
0 & \text{if } 1 < B.
\end{cases}$$

Note that $\tilde{b} = \sqrt{1 - 2B}$ and $\tilde{b} = \sqrt{1 + 2B}$, respectively.

In Figure 1, $E[R^w(\cdot)]$ is absolutely positive for any buy price $B \in (0, 1)$. Moreover, $E[R^w(\cdot)]$ is maximized at some buy price $B \in \left[\frac{1}{2}, 1\right]$. Also, in the interval $\left[\frac{1}{2}, 1\right]$, $E[R^w(\cdot)]$ is monotone decreasing with respect to the buy price $B$.

We reconsider second-price, sealed-bid auctions without a buy price $B$. In these auctions, introducing a reserve price is the most straightforward way for sellers to guarantee a positive equilibrium revenue. Here let us denote by $E[R^r(\cdot)]$ an equilibrium revenue in second-price, sealed-bid auctions without a buy price $B$ but with a reserve price $r \in [0, \omega]$. We then have a similar result to that of Proposition 1.

**Proposition 4.** Consider second-price, sealed-bid auctions with a reserve price $r \in [0, \omega]$. Then, the set of the equilibrium revenues is

$$[E[R^r(\omega)], E[R^r(r)]].$$
Proof. See Appendix.

If a seller sets a reserve price $r$, then the domain of available bids is restricted. In a result, she can obtain a positive equilibrium revenue. And the seller acquires the maximum equilibrium revenue at a symmetric strategy equilibrium.

As well as the case of second-price, sealed-bid auctions with a buy price $B$, we focus on the minimum of the set of the equilibrium revenues of each second-price, sealed-bid auction with a reserve price $r$. We can regard it as a function of the reserve price $r$. Here let us denote by $E[R^w_0(r)]$ the minimum of the set of the equilibrium revenues in a second-price, sealed-bid auction with the reserve price $r$.

**Example 2.** Suppose that bidders’ types are uniformly distributed on the interval $[0,1]$. Then, the minimum of the set of the equilibrium revenues is

$$E[R^w_0(r)] = r - r^3 \quad \text{if } 0 \leq r \leq \omega.$$

In Figure 2, both $E[R^w(B)]$ and $E[R^w_0(r)]$ are depicted. To guarantee an equilibrium revenue that is greater than the maximum of $E[R^w(B)]$, a seller can choose a reserve price $r$ from a wide interval.

6 Conclusion

We have investigated how the introduction of a buy price affects sellers’ equilibrium revenues. In the analysis, we have looked at not only a truth-telling strategy equilibrium but also elaborate pure strategy equilibria. For each buy price auction, we have characterized the set of the equilibrium revenues. We have shown that by introducing any effective buy price,
sellers can always obtain a positive equilibrium revenue. In second-price, sealed-bid auctions without a buy price, on the other hand, the minimum of the set of the equilibrium revenues is zero. Therefore, sellers can avoid a risk that they obtain zero equilibrium revenue by introducing a buy price. Our results provide a new reason why sellers so often introduce a buy price in Internet auctions.

Appendix

Proof of Proposition 1

We organize the proof in two steps.

Step 1. We show that each element in the interval $[0, \frac{\omega}{3}]$ is an equilibrium revenue.

Let us consider the strategy profile $\sigma = (\sigma_i(\cdot), \sigma_j(\cdot))$ such that

$$
\sigma_i(t_i) = \begin{cases} 
0 & \text{if } 0 \leq t_i \leq b, \\
t_i & \text{if } b < t_i \leq \omega 
\end{cases}
$$

and

$$
\sigma_j(t_j) = \begin{cases} 
b & \text{if } 0 \leq t_j \leq b, \\
t_j & \text{if } b < t_j \leq \omega.
\end{cases}
$$

Note that $b \in [0, \omega]$.

Fix $b \in [0, \omega]$ arbitrarily. The corresponding strategy profile $\sigma$ is determined. We show
that the strategy profile $\sigma$ is an equilibrium. Consider the incentive constraints of bidder $i$. For all $t_i \in [0, b]$, he obtains at most zero expected payoff by submitting other bid, for example the bid $b$. For all $t_i \in (b, \omega]$, he only reduces his expected payoff even if he submits a bid that is greater or less than his type $t_i$. Next, consider the incentive constraints of bidder $j$. For all $t_j \in [0, b]$, he only reduces his expected payoff even if he submits a bid that is greater or less than the bid $b$. And, for all $t_j \in (b, \omega]$, he cannot improve his expected payoff by submitting a bid that is greater than his type $t_j$. Therefore, the strategy profile $\sigma$ is an equilibrium.

At the equilibrium $\sigma$, the seller receives the expected revenue

$$
\int_b^\omega \left( \int_{t_i}^\omega t_i f(t_j) dt_j \right) f(t_i) dt_i + \int_b^b \left( \int_b^\omega b f(t_i) dt_i \right) f(t_j) dt_j + \int_b^\omega \left( \int_{t_j}^\omega t_j f(t_i) dt_i \right) f(t_j) dt_j
$$

$$
= \omega^3 - b^3 \over 3 \omega^2.
$$

By the way of construction of the strategy profile $\sigma$, therefore, each element in the interval $[0, {\omega \over 3})$ is an equilibrium revenue.

**Step 2.** We show that the equilibrium revenue $\omega^3 / 3$ is the maximum of the set of the equilibrium revenues.

First, to be an equilibrium, any strategy profile must satisfy the following property. Suppose that at least one bidder’s types do not submit his own types. At any equilibrium, there are only type intervals $[t_i, t_j]$ in which one bidder’s all types in the interval $[t_i, t_j]$ submit the bid $t_i$, and the other bidder’s all types in the interval $[t_i, t_j]$ submit the bid $t_j$. Suppose by contradiction that a strategy profile which does not satisfy the property is an equilibrium.  

We can find multiple intervals between the bids of one bidder $i$’s types $t_i$ in the interval $[0, \omega]$ and his own types $t_i$, that is, $(t_i, b_i]$ (or $[b_i, t_i]$), where the other bidder $j$’s multiple types submit bids. Other things being equal, bidder $i$ can improve his expected payoff by submitting his own types $t_i$, which follows the contradiction.

We consider whether there exists an equilibrium that yields a greater equilibrium revenue than that of the truth-telling strategy equilibrium. We then consider equilibria in which one bidder’s all types in the interval $[t_i, t_j]$ submit the bid $t_i$, the other bidder’s all types in the interval $[t_i, t_j]$ submit the bid $t_j$. This is because we allow bidders to play a strategy that has one plateau.

\[\text{12}^\text{In Blume and Heidhues (2001), it is shown that strategy profiles like the strategy profile $\sigma$ are an equilibrium.}\]

\[\text{13}^\text{We do not consider strategy profiles where bidders behave in a manner that violates the property with zero probability.}\]
Without loss of generality, let us consider the strategy profile \( \sigma = (\sigma_i(\cdot), \sigma_j(\cdot)) \) such that

\[
\sigma_i(t_i) = \begin{cases} 
  t_i & \text{if } 0 \leq t_i \leq \bar{t}, \\
  \tilde{t} & \text{if } \frac{3}{2} < t_i \leq \bar{t}, \\
  t_i & \text{if } \frac{3}{2} < t_i \leq \omega 
\end{cases}
\]

and

\[
\sigma_j(t_j) = \begin{cases} 
  t_j & \text{if } 0 \leq t_j \leq \bar{t}, \\
  \tilde{t} & \text{if } \frac{3}{2} < t_j \leq \bar{t}, \\
  t_j & \text{if } \frac{3}{2} < t_j \leq \omega 
\end{cases}
\]

From the above argument, the strategy profile \( \sigma \) is an equilibrium.

Here let us denote by \( E[R^*] \) and \( E[\bar{R}] \) the equilibrium revenue of the truth-telling strategy profile and the strategy profile \( \sigma \), respectively. We can find clear differences in seller’s expected revenues for three cases: (i) both bidders’ types are in the interval \([\bar{t}, \tilde{t}]\). (ii) bidder \( i \)’s type is in the interval \([\bar{t}, \tilde{t}]\) and bidder \( j \)’s type is not less than \( \tilde{t} \). (iii) bidder \( i \)’s type is not less than \( \tilde{t} \) and bidder \( j \)’s type is in the interval \([\bar{t}, \tilde{t}]\). Note that winners of the cases (ii) and (iii) do not change, respectively. Then, we have

\[
E[R^*] - E[\bar{R}] = \int_{\bar{t}}^{\tilde{t}} \left( \int_{\bar{t}}^{\tilde{t}} t_j f(t_i) dt_i \right)f(t_j) dt_j + \int_{\bar{t}}^{\tilde{t}} \left( \int_{\bar{t}}^{\tilde{t}} t_i f(t_j) dt_j \right)f(t_i) dt_i
\]

\[
- \int_{\bar{t}}^{\tilde{t}} \left( \int_{\frac{3}{2}}^{\tilde{t}} f(t_i) dt_i \right) f(t_j) dt_j - \int_{\bar{t}}^{\tilde{t}} \left( \int_{\frac{3}{2}}^{\tilde{t}} f(t_j) dt_j \right) f(t_i) dt_i
\]

\[
= \frac{(\bar{t} - \tilde{t})^2}{3\omega^2} > 0.
\]

Therefore, \( E[\bar{R}] \) is less than \( E[R^*] \). In other words, \( \frac{\omega}{3} \) is the maximum of the set of the equilibrium revenues.

**A second-price, sealed-bid auction with the buy price \( B = 0 \)**

We consider a second-price, sealed-bid auction with the buy price \( B = 0 \). In this auction, a strategy profile in which both bidders’ all types submit the buy price \( B \) is a unique equilibrium. Therefore, the set of the equilibrium revenues is \( \{B\} \).

**Second-price, sealed-bid auctions with a buy price \( B \in (\omega, +\infty) \)**

We consider second-price, sealed-bid auctions with a buy price \( B \in (\omega, +\infty) \). Since the buy price \( B \) is greater than the highest possible valuation, no one bids the buy price \( B \) at any equilibrium. Therefore, the set of the equilibrium revenues is the same as that of second-price, sealed-bid auctions without a buy price \( B \).
Proof of Claim 1

Suppose that both bidders’ all types which are not less than the buy price $B$ submit it. Without loss of generality, consider the incentive constraints of bidder $i$. We wonder whether for all $t_i \in [B, \omega]$,

$$
\int_0^\theta (t_i - B)f(t_j)dt_j + \frac{1}{2} \int_B^\omega (t_i - B)f(t_j)dt_j \geq \int_0^\theta (t_i - 0)f(t_j)dt_j
$$

(1)
holds. RHS of (1) is the most profitable deviation that bidder $i$ can obtain among all strategy profiles such that both bidders’ all types which are not less than the buy price $B$ submit it. Calculating (1), we have

$$
\frac{1}{2\omega}\{(\omega - B)t_i + B^2 - \omega B - 2\theta B + \theta^2 \geq 0\}.
$$

Substituting $B$ for $\theta$, we have

$$
\frac{1}{2\omega}\{(\omega - B)t_i - \omega B \geq 0\}.
$$

Here let

$$
\phi(t_i) \equiv (\omega - B)t_i - \omega B.
$$

For $t_i \in [B, \omega]$, the function $\phi(\cdot)$ is increasing with respect to $t_i$. To be an equilibrium, it is sufficient to show that (1) holds for $t_i = B$. However,

$$
-B^2 \geq 0
$$
does not hold.

From the above argument, the incentive constraints of at least one bidder’s all types which are not less than the buy price $B$ do not hold at any strategy profiles such that both bidders’ all types which are not less than the buy price $B$ submit it.

Proof of Proposition 2

We organize the proof in three steps.

Step 1. We show that a strategy profile is an equilibrium.
Consider the following strategy profile $\sigma^H = (\sigma_i^H(\cdot), \sigma_j^H(\cdot))$ such that

$$
\sigma_i^H(t_i) = \begin{cases} 
  t_i & \text{if } 0 \leq t_i \leq \underline{t}, \\
  \underline{t} & \text{if } \underline{t} < t_i \leq \overline{t}, \\
  t_i & \text{if } \overline{t} < t_i \leq \overline{b}, \\
  \overline{b} & \text{if } B \leq t_i \leq \underline{t}_i, \\
  B & \text{if } \underline{t}_i < t_i \leq \omega 
\end{cases}
$$

and

$$
\sigma_j^H(t_j) = \begin{cases} 
  t_j & \text{if } 0 \leq t_j \leq \underline{t}, \\
  \underline{t} & \text{if } \underline{t} < t_j \leq \overline{t}, \\
  t_j & \text{if } \overline{t} < t_j \leq \overline{b}, \\
  \overline{b} & \text{if } B \leq t_j \leq \underline{t}_j, \\
  B & \text{if } \underline{t}_j < t_j \leq \omega 
\end{cases}
$$

where $0 \leq \underline{t} \leq \overline{t} \leq \overline{b}$.

We show that the strategy profile $\sigma^H$ is a Bayesian Nash equilibrium. Consider the incentive constraints of bidder $i$. For all $t_i \in [0, \underline{t}]$, he reduces his expected payoff by submitting a bid that is greater or less than his type $t_i$. For all $t_i \in (\underline{t}, \overline{t}]$, he obtains at most the same expected payoff even by submitting other bids. For all $t_i \in (\overline{t}, \overline{b}]$, he reduces his expected payoff by submitting a bid that is greater or less than his type $t_i$. For all $t_i \in [B, \underline{t}_i]$,

$$
\int_0^\underline{t} (t_i - t_j) f(t_j) dt_j + \int_{\underline{t}}^{\overline{t}} (t_i - \overline{t}) f(t_j) dt_j + \int_{\overline{t}}^{\overline{b}} (t_i - t_j) f(t_j) dt_j + \frac{1}{2} \int_B^{\underline{t}_i} (t_i - \overline{b}) f(t_j) dt_j \\
\geq \int_0^{\overline{b}} (t_i - B) f(t_j) dt_j + \frac{1}{2} \int_{\underline{t}_i}^{\overline{b}} (t_i - B) f(t_j) dt_j 
$$

must hold. For all $t_i \in (\overline{t}_i, \omega]$,

$$
\int_0^{\overline{b}} (t_i - B) f(t_j) dt_j + \frac{1}{2} \int_{\overline{b}}^{\omega} (t_i - B) f(t_j) dt_j \\
\geq \int_0^{\overline{b}} (t_i - t_j) f(t_j) dt_j + \int_{\overline{b}}^{\overline{t}} (t_i - \overline{t}) f(t_j) dt_j + \int_{\overline{t}}^{\overline{b}} (t_i - t_j) f(t_j) dt_j + \frac{1}{2} \int_B^{\omega} (t_i - \overline{b}) f(t_j) dt_j 
$$
must hold. Therefore, the systems of inequalities (2) and (3) are satisfied if and only if

\[
\int_0^L (t_i^b - t_j) f(t_j) dt_j + \int_L^\tau (t_i^b - \bar{t}) f(t_j) dt_j + \int_\tau^\bar{b} (t_i^b - t_j) f(t_j) dt_j + \frac{1}{2} \int_B^{t_i^b} (t_i^b - \bar{b}) f(t_j) dt_j \\
= \int_0^{t_i^b} (t_i^b - B) f(t_j) dt_j + \frac{1}{2} \int_{t_i^b}^{\omega} (t_i^b - B) f(t_j) dt_j
\]

(4)

holds. This is because a function

\[
\int_0^L (x - t_j) f(t_j) dt_j + \int_L^\tau (x - \bar{t}) f(t_j) dt_j + \int_\tau^\bar{b} (x - t_j) f(t_j) dt_j + \frac{1}{2} \int_B^{t_i^b} (x - \bar{b}) f(t_j) dt_j \\
- \int_0^{t_i^b} (x - B) f(t_j) dt_j + \frac{1}{2} \int_{t_i^b}^{\omega} (x - B) f(t_j) dt_j
\]

is linear with respect to \(x\).

Next, consider the incentive constraints of bidder \(j\). For all \(t_j \in [0, \bar{t}]\), he reduces his expected payoff by submitting a bid that is greater or less than his type \(t_j\). For all \(t_j \in (\bar{t}, \bar{b}]\), he obtains at most the same expected payoff by submitting other bids. For all \(t_j \in [\bar{b}, B]\), he reduces his expected payoff by submitting a bid that is greater or less than his type \(t_j\). For all \(t_j \in [B, t_i^b]\),

\[
\int_0^L (t_j - t_i) f(t_i) dt_i + \int_L^\tau (t_j - \bar{t}) f(t_i) dt_i + \int_\tau^{\bar{t}} (t_j - t_i) f(t_i) dt_i + \frac{1}{2} \int_B^{t_i^b} (t_j - \bar{b}) f(t_i) dt_i \\
\geq \int_0^{t_i^b} (t_j - B) f(t_i) dt_i + \frac{1}{2} \int_{t_i^b}^{\omega} (t_j - B) f(t_i) dt_i
\]

(5)

must hold. For all \(t_j \in (t_i^b, \omega]\),

\[
\int_0^{t_i^b} (t_j - B) f(t_i) dt_i + \frac{1}{2} \int_{t_i^b}^{\omega} (t_j - B) f(t_i) dt_i \\
\geq \int_0^L (t_j - t_i) f(t_i) dt_i + \int_L^\tau (t_j - \bar{t}) f(t_i) dt_i + \int_\tau^{\bar{t}} (t_j - t_i) f(t_i) dt_i + \frac{1}{2} \int_B^{t_i^b} (t_j - \bar{b}) f(t_i) dt_i
\]

(6)

must hold. Therefore, the systems of inequalities (5) and (6) are satisfied if and only if

\[
\int_0^L (t_j^b - t_i) f(t_i) dt_i + \int_L^\tau (t_j^b - \bar{t}) f(t_i) dt_i + \int_\tau^{\bar{b}} (t_j^b - t_i) f(t_i) dt_i + \frac{1}{2} \int_B^{t_i^b} (t_j^b - \bar{b}) f(t_i) dt_i \\
= \int_0^{t_i^b} (t_j^b - B) f(t_i) dt_i + \frac{1}{2} \int_{t_i^b}^{\omega} (t_j^b - B) f(t_i) dt_i
\]

(7)
holds. By calculating (4) and (7), respectively, we have

\[ t_i^b = \frac{\omega B - (\bar{t} - t)^2}{\omega - B} \]

and

\[ t_j^b = \frac{\omega B + (\bar{t} - t)^2}{\omega - B}. \]

**Step 2.** We show that it is enough to consider the strategy profile \( \sigma^2. \)

At the equilibrium \( \sigma^H, \) the seller receives an expected revenue

\[
\begin{align*}
\int_0^\ell \left( \int_{t_i}^{t_i^b} t_i f(t_j) dt_j \right) f(t_i) dt_i + \int_0^\ell \left( \int_{t_i}^{t_i^b} t f(t_j) dt_j \right) f(t_i) dt_i + \int_0^\ell \left( \int_{t_i}^{t_i^b} t_i f(t_j) dt_j \right) f(t_i) dt_i \\
+ \int_{B}^{t_i^b} \left( \int_{B}^{t_i^b} B f(t_j) dt_j \right) f(t_i) dt_i + \int_0^\ell \left( \int_{t_i}^{t_i^b} B f(t_j) dt_j \right) f(t_i) dt_i + \int_0^\ell \left( \int_{t_i}^{t_i^b} B f(t_j) dt_j \right) f(t_i) dt_i \\
+ \int_0^\ell \left( \int_{t_i}^{t_i^b} t_j f(t_i) dt_i \right) f(t_j) dt_j + \int_0^\ell \left( \int_{t_i}^{t_i^b} t f(t_i) dt_i \right) f(t_j) dt_j + \int_0^\ell \left( \int_{t_i}^{t_i^b} t_i f(t_i) dt_i \right) f(t_j) dt_j \\
+ \int_0^\ell \left( \int_{t_i}^{t_i^b} B f(t_j) dt_j \right) f(t_i) dt_i \end{align*}
\]

\[
= \frac{1}{6\omega^2} \left\{ 2t_i^3 + 3(t_i^b - t_j^b) t_i^2 - 6(t_i^b - t_j^b) t_i^3 - 6t_i^2 t_i^3 + 6t_i^2 t_i^3 + 3(t_i^b - t_j^b) t_i^3 - 2t_i^3 \\
+ 3\bar{t}^2 t_i^b - 6\bar{t} t_i^b - 6\bar{t} t_j^b - 3\bar{t}^2 t_j^b - 6\bar{t} t_i^b - 6\bar{t} t_j^b + 4\bar{t}^3 - 6\bar{t}^2 B + 6\bar{t}^2 B - 6\bar{t}^2 B - 6\bar{t}^2 B + 4\bar{t}^3 + 6\bar{t}^2 B + 6\omega^2 B \right\}. \]

Evaluating \( \bar{t} \) at \( B, \) we have

\[
\frac{1}{6\omega^2} \left\{ 2t_i^3 + 3(t_i^b - t_j^b) t_i^2 - 6(t_i^b - t_j^b) t_i^3 - 6t_i^2 t_i^3 + 6t_i^2 t_i^3 + 3(t_i^b - t_j^b) t_i^3 - 2t_i^3 + 2B^3 - 3(t_i^b + t_j^b) B + 6\omega^2 B \right\}. \]

Here let us denote by \( E[R_i^B] \) the seller’s expected revenue.

Partially differentiating \( E[R_i^B] \) with respect to \( \bar{t}, \) we have

\[
\frac{\partial E[R_i^B]}{\partial \bar{t}} = \frac{1}{6\omega^2} \left[ (2t_i^3 + 3(t_i^b - t_j^b) t_i^2 - 6(t_i^b - t_j^b) t_i^3 - 6t_i^2 t_i^3 + 6t_i^2 t_i^3 + 3(t_i^b - t_j^b) t_i^3 - 2t_i^3 + 2B^3 - 3(t_i^b + t_j^b) B + 6\omega^2 B) \right] \]

\[
= \frac{1}{6\omega^2} \left[ (2t_i^3 + 3(t_i^b - t_j^b) t_i^2 - 6(t_i^b - t_j^b) t_i^3 - 6t_i^2 t_i^3 + 6t_i^2 t_i^3 + 3(t_i^b - t_j^b) t_i^3 - 2t_i^3 + 2B^3 - 3(t_i^b + t_j^b) B + 6\omega^2 B) \right] \]

\[
\times \left[ (t_i^b)^2 + \frac{2(\bar{t} - t)}{\omega - B} \right]. \]

Substituting

\[
(t_i^b)^2 = \frac{2(\bar{t} - t)}{\omega - B}, \]

\[
\text{we have } E[R_i^B] = 17. \]
and

\[ (t_j^b)'_t = \frac{-2(t - \overline{t})}{\omega - B} \]

into (8), we have

\[ \frac{\partial E[R^B]}{\partial t} = -\frac{(\overline{t} - t)^2}{\omega^2(\omega - B)} \{4(\overline{t} - t) + (\omega - B)\} < 0. \]

Similarly, partially differentiating \( E[R^B] \) with respect to \( t \), we have

\[
\frac{\partial E[R^B]}{\partial t} = \frac{1}{6\omega^2} \left( 6\hat{t} t_j^b + 6t_j^b'(t_j^b)'_t - 6t_j^b 3t_j^b(t_j^b)'_t - 12\hat{t} + 6t_j^b 3\hat{t}^2(t_j^b)'_t - 3(t_j^b)'_t B^2 \right.
\]

\[ - 3(t_j^b)'_t B^2 + 6t_j^b + 3t_j^b(t_j^b)'_t + 3\hat{t}^2(t_j^b)'_t - 6t_j^b 6\hat{t}(t_j^b)'_t + 6\hat{t}^2. \] (9)

Substituting

\[ (t_j^b)'_t = \frac{-2(\overline{t} - t)}{\omega - B} \]

and

\[ (t_j^b)'_\overline{t} = \frac{2(\overline{t} - t)}{\omega - B} \]

into (9), we have

\[ \frac{\partial E[R^B]}{\partial t} = \frac{(\overline{t} - t)^2}{\omega^2(\omega - B)} \{4(\overline{t} - t) + (\omega - B)\} > 0. \]

From the above argument, \( E[R^B] \) is minimized at \( (t, \overline{t}) = (0, \overline{b}) \), and maximized at \( (\overline{t}, \overline{t}) = (t, t) \) for all \( t \in [0, \overline{b}] \). These facts imply that it is enough to consider the strategy profile \( \sigma^2 \).

**Step 3.** We derive the interval of the bid \( b \).

We can rewrite both \( t_i^b \) and \( t_j^b \) for the strategy profile \( \sigma^2 \) as follows.

\[ t_i^b = \frac{\omega B - b^2}{\omega - B} \]

and

\[ t_j^b = \frac{\omega B + b^2}{\omega - B}. \]

We need to check that both \( t_i^b \) and \( t_j^b \) are well-defined for each \( b \in [0, \overline{b}] \). First we consider \( t_i^b \). If \( b = \overline{b} \), we have

\[ t_i^b = \frac{\omega B - \overline{b}^2}{\omega - B}. \]
Evaluating \( b \) at \( B \), we have
\[ t^b_i = B. \]
As \( t^b_i \) is decreasing with respect to \( b \), \( t^b_i \) is always greater than \( B \) for each \( b \in [0, \bar{b}] \). If \( b = 0 \), on the other hand, we have
\[ t^0_i = \frac{\omega B}{\omega - B}, \]
which is less than \( \omega \). Therefore, \( t^b_i \) is always less than \( \omega \) for each \( b \in [0, \bar{b}] \).

Now, we consider \( t^b_j \). To be less than \( \omega \),
\[ t^b_j = \frac{\omega B + b^2}{\omega - B} < \omega \iff b < \sqrt{\omega^2 - 2\omega B} \]
must hold. Note that \( b \geq 0 \). Moreover, we need to check that \( \bar{b} \leq \sqrt{\omega^2 - 2\omega B} \). Raising both sides to the second power, we have
\[ \bar{b}^2 \leq \omega^2 - 2\omega B. \]
Evaluating \( b \) at \( B \), we have
\[ B^2 \leq \omega^2 - 2\omega B. \]
Therefore, we can choose \( b \in [0, \bar{b}] \) arbitrarily if \( B \leq (\sqrt{2} - 1)\omega \). And, we can only choose \( b \) from the interval \( [0, \sqrt{\omega^2 - 2\omega B}] \) if \( B > (\sqrt{2} - 1)\omega \).

**Proof of Proposition 3**

We organize the proof in two steps.

*Step 1.* We show that a strategy profile is an equilibrium.

Consider the strategy profile \( \sigma^N = (\sigma^N_i(\cdot), \sigma^N_j(\cdot)) \) such that
\[
\sigma^N_i(t_i) = \begin{cases} 
  t_i & \text{if } 0 \leq t_i \leq \bar{t}, \\
  \bar{t} & \text{if } \bar{t} < t_i \leq \bar{t}, \\
  t_i & \text{if } \bar{t} < t_i \leq \bar{b}, \\
  \bar{b} & \text{if } B \leq t_i \leq \omega \n\end{cases}
\]
and
\[
\sigma^N_j(t_j) = \begin{cases} 
  t_j & \text{if } 0 \leq t_j \leq \bar{t}, \\
  \bar{t} & \text{if } \bar{t} < t_j \leq \bar{t}, \\
  t_j & \text{if } \bar{t} < t_j \leq \bar{b}, \\
  \bar{b} & \text{if } B \leq t_j \leq \omega \n\end{cases}
\]
where \( 0 \leq t \leq \bar{t} \leq \bar{\bar{t}} \).

We show that the strategy profile \( \sigma^N \) is a Bayesian Nash equilibrium. Consider the incentive constraints of bidder \( i \). For all \( t_i \in [0, \bar{t}] \), he reduces his expected payoff by submitting a bid that is greater or less than his type \( t_i \). For all \( t_i \in (\bar{t}, \bar{\bar{t}}] \), he reduces his expected payoff by submitting the bid \( \bar{t} \). For all \( t_i \in (\bar{\bar{t}}, \bar{\bar{w}}] \), he reduces his expected payoff by submitting a bid that is greater or less than his type \( t_i \). For all \( t_i \in [B, \bar{w}] \), we need to consider what the most profitable deviation is. Suppose that

\[
\int_0^\omega (t_i - B) f(t_j) dt_j \geq \lim_{\varepsilon \to 0} \left\{ \int_0^{\bar{t}} (t_i - t_j) f(t_j) dt_j + \int_{\bar{t}}^{\bar{\bar{t}}} (t_i - t_j) f(t_j) dt_j + \int_{\bar{\bar{t}}}^{\omega} (t_i - t_j) f(t_j) dt_j \right\}
\]

(10)

Calculating (10), we have

\[
\frac{1}{\omega} \left\{ (\omega - \bar{b}) t_i - \omega B + \frac{\bar{b}^2}{2} + \frac{(\bar{t} - t)^2}{2} \geq 0 \right\}.
\]

Evaluating \( \bar{b} \) at \( B \), we have

\[
\frac{1}{\omega} \left\{ (\omega - B) t_i - \omega B + \frac{B^2}{2} - \frac{(\bar{t} - t)^2}{2} \geq 0 \right\}.
\]

Here let

\[
\gamma(t_i) \equiv (\omega - B) t_i - \omega B + \frac{B^2}{2} - \frac{(\bar{t} - t)^2}{2}.
\]

For \( t_i \in [B, \bar{w}] \), the function \( \gamma(t_i) \) is increasing with respect to \( t_i \). In addition, \( \gamma(B) < 0 \).

Therefore, for all \( t_i \in [B, \bar{t}^*] \),

\[
\int_0^{\bar{t}} (t_i - t_j) f(t_j) dt_j + \int_{\bar{t}}^{\bar{\bar{t}}} (t_i - t_j) f(t_j) dt_j + \frac{1}{2} \int_{\bar{\bar{t}}}^{\omega} (t_i - \bar{b}) f(t_j) dt_j
\]

(11)

\[
\geq \int_0^{\bar{t}} (t_i - t_j) f(t_j) dt_j + \int_{\bar{t}}^{\bar{\bar{t}}} (t_i - t_j) f(t_j) dt_j + \frac{1}{2} \int_{\bar{\bar{t}}}^{\omega} (t_i - t_j) f(t_j) dt_j
\]

and for all \( t_i \in (\bar{t}^*, \bar{\bar{w}}] \),

\[
\int_0^{\bar{t}} (t_i - t_j) f(t_j) dt_j + \int_{\bar{t}}^{\bar{\bar{t}}} (t_i - t_j) f(t_j) dt_j + \frac{1}{2} \int_{\bar{\bar{t}}}^{\omega} (t_i - \bar{b}) f(t_j) dt_j
\]

(12)

must hold. Here let \( t_i^* = \{ 2\omega B - B^2 - (\bar{t} - t)^2 \} / 2(\omega - B) \).

Clearly, (11) holds for each \( t_i \in [B, \bar{t}^*] \). It is sufficient to consider whether (12) holds or
not. Calculating (12), we have
\[
\frac{1}{2\omega} \{ -\omega + B - 2\bar{b} - (\bar{t} - \bar{t})^2 - \bar{\omega} + \bar{b} B + 2\omega B \geq 0 \}.
\]
Evaluating \( \frac{5}{5} \) at \( B \), we have
\[
\frac{1}{2\omega} \{ -\omega + B - (\bar{t} - \bar{t})^2 + \omega B \geq 0 \}.
\]
Here let
\[
\iota(t_i) \equiv -\omega + B - (\bar{t} - \bar{t})^2 + \omega B.
\]
For \( t_i \in [B, t_i^*] \), \( \iota(\cdot) \) is decreasing with respect to \( t_i \). Therefore, it is sufficient to show that (12) holds for \( t_i = \omega \). That is,
\[
B \geq \frac{(\bar{t} - \bar{t})^2 + \omega^2}{2\omega}. \tag{13}
\]
Now, we consider whether \( t_i^* \) is well-defined. Clearly, \( t_i^* > B \). Therefore, we consider whether
\[
\frac{2\omega B - B^2 - (\bar{t} - \bar{t})^2}{2(\omega - B)} \leq \omega. \tag{14}
\]
Calculating (14) with respect to \( B \), we have
\[
B \leq 2\omega - \sqrt{2\omega^2 - (\bar{t} - \bar{t})^2} \quad \text{and} \quad 2\omega + \sqrt{2\omega^2 - (\bar{t} - \bar{t})^2} \leq B.
\]
Clearly, \( 2\omega + \sqrt{2\omega^2 - (\bar{t} - \bar{t})^2} > \omega \), we have
\[
B \leq 2\omega - \sqrt{2\omega^2 - (\bar{t} - \bar{t})^2}. \tag{15}
\]
Next, consider the incentive constraints of bidder \( j \). For all \( t_j \in [0, \bar{t}] \), he reduces his expected payoff by submitting a bid that is greater or less than his type \( t_j \). For all \( t_j \in (\bar{t}, \bar{b}] \), he reduces his expected payoff by submitting the bid \( \bar{t} \). For all \( t_j \in (\bar{t}, B] \), he reduces his expected payoff by submitting a bid that is greater or less than his type \( t_j \). For all \( t_i \in [B, \omega] \), we need to consider what the most profitable deviation is. Suppose that
\[
\int_{0}^{t_j} (t_j - B) f(t_i) dt_i \geq \lim_{\varepsilon \to 0} \left\{ \int_{0}^{\bar{t}} (t_j - t_i) f(t_i) dt_i + \int_{t_i}^{\bar{t}} (t_j - t_i) f(t_i) dt_i + \int_{t_i}^{\bar{b} - \varepsilon} (t_j - t_i) f(t_i) dt_i \right\} \tag{16}
\]
If (10) does not hold, then (16) does not hold, neither. We then assume that (10) holds. If (16) does not hold, it is enough to consider (12). If (16) hold, then
\[
\int_{0}^{t_j} (t_j - t_i) f(t_i) dt_i + \int_{t_i}^{\bar{t}} (t_j - t_i) f(t_i) dt_i + \int_{t_i}^{\bar{b}} (t_j - t_i) f(t_i) dt_i + \frac{1}{2} \int_{B}^{\bar{b}} (t_j - B) f(t_i) dt_i \geq \int_{0}^{\omega} (t_j - B) f(t_i) dt_i \tag{17}
\]
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must hold. Calculating (17), we have
\[
\frac{1}{2\omega} \{ -(\omega + B - 2\bar{t})t_j + (\bar{t} - t)^2 - \bar{t}^2 - \bar{t}\omega + \bar{t}B + 2\omega B \geq 0 \}.
\]
Evaluating \( \mathbf{5} \) at \( B \), we have
\[
\frac{1}{2\omega} \{ -(\omega - B)t_j + (\bar{t} - t)^2 + \omega B \geq 0 \}.
\]
Here let
\[
\kappa(t_j) \equiv -(\omega - B)t_j + (\bar{t} - t)^2 + \omega B.
\]
For \( t_j \in [B, \omega] \), \( \kappa(\cdot) \) is decreasing with respect to \( t_j \). Therefore, it is sufficient to show that (17) holds for \( t_j = \omega \). That is,
\[
B \geq \frac{(\bar{t} - t)^2 + \omega^2}{2\omega}.
\]
By (13), (15), and (18), the strategy profile \( \sigma^N \) is an equilibrium if
\[
\frac{(\bar{t} - t)^2 + \omega^2}{2\omega} \leq B \leq 2\omega - \sqrt{2\omega^2 - (\bar{t} - t)^2}.
\]
Note that
\[
2\omega - \sqrt{2\omega^2 - (\bar{t} - t)^2} - \left\{ \frac{(\bar{t} - t)^2 + \omega^2}{2\omega} \right\} = \frac{\sqrt{2\omega^2 - (\bar{t} - t)^2} - \omega^2}{2\omega} \geq 0.
\]
In the case that (15) does not hold, (11) holds for all \( t_i \in [B, \omega] \). Therefore, it is enough to consider (18), which is clearly satisfied because \( B \geq \frac{\omega}{2} \).

**Step 2.** We show that it is enough to consider the strategy profile \( \sigma^i \).

At the equilibrium \( \sigma^N \), the seller receives an expected revenue
\[
\int_0^{\bar{t}} \left( \int_0^\omega t_i f(t_j)dt_j \right) f(t_i)dt_i + \int_\omega^{\bar{t}} \left( \int_0^\omega t_i f(t_j)dt_j \right) f(t_i)dt_i + \int_\bar{t}^\omega \left( \int_0^\omega t_i f(t_j)dt_j \right) f(t_i)dt_i + \int_0^{\bar{t}} \left( \int_0^\omega t_i f(t_j)dt_j \right) f(t_i)dt_i + \int_\omega^{\bar{t}} \left( \int_0^\omega t_i f(t_j)dt_j \right) f(t_i)dt_j + \int_\bar{t}^\omega \left( \int_0^\omega t_i f(t_j)dt_j \right) f(t_i)dt_j + \int_0^{\bar{t}} \left( \int_0^\omega t_i f(t_j)dt_j \right) f(t_i)dt_i + \int_\omega^{\bar{t}} \left( \int_0^\omega t_i f(t_j)dt_j \right) f(t_i)dt_j + \int_\bar{t}^\omega \left( \int_0^\omega t_i f(t_j)dt_j \right) f(t_i)dt_j
\]
\[
= \frac{1}{6\omega^2} \left\{ -4\bar{t}^3 + 6\bar{t}\omega^2 + 6\bar{t}\omega^2 - 12\omega\bar{t}\omega B + 6\bar{t}B^2 - 2(\bar{t} - t)^3 \right\}.
\]
Evaluating $\tilde{b}$ at $B$, we have

$$\frac{1}{3\omega^2} \{3\omega^2 B - 3\omega B^2 + B^3 - (\tilde{t} - t)^3\}.$$

Here let us denote by $E[R_{II}^B]$ the seller’s expected revenue. Clearly, $E[R_{II}^B]$ is decreasing with respect to $\tilde{t}$, and increasing with respect to $t$. Therefore, $E[R_{II}^B]$ is minimized at $(\tilde{t}, \tilde{t}) = (0, \tilde{b})$, and maximized at $(\tilde{t}, \tilde{t}) = (t, t)$ for all $t \in [0, \tilde{b}]$. These facts imply that it is enough to consider the strategy profile $\sigma^\delta$.

**Proof of Theorem 1**

By Propositions 2 and 3, the minimum of the set of the equilibrium revenues is

- $E[R_f^B(\tilde{b})]$ if $0 \leq B < (\sqrt{2} - 1)\omega$,
- $E[R_f^B(\tilde{b})]$ if $(\sqrt{2} - 1)\omega \leq B < \frac{\omega}{2}$,
- $E[R_{II}^B(\tilde{b})]$ if $\frac{\omega}{2} \leq B \leq \omega$.

Under the same buy price $B$, $E[R_f^B(\tilde{b})] \leq E[R_f^B(\hat{b})]$. Therefore, we consider $E[R_f^B(\tilde{b})]$ instead of $E[R_f^B(\hat{b})]$ for the buy price $B \in [(\sqrt{2} - 1)\omega, \frac{\omega}{2})$. Calculating $E[R_f^B(\tilde{b})]$, we have

$$E[R_f^B(\tilde{b})] = \frac{B}{\omega^2(\omega - B)}(\omega^3 - \omega^2 B - \omega B^2 - B^3).$$

Note that we evaluate $\tilde{b}$ at $B$. Differentiating $\omega^3 - \omega^2 B - \omega B^2 - B^3$ with respect to $B$, we have

$$-3 \left( B + \frac{\omega}{3} \right)^2 - \frac{2}{3} \omega^2 < 0.$$

Therefore, $E[R_f^B(\tilde{b})]$ is minimized at $B = \frac{\omega}{2}$. In this case, we have

$$E[R_f^B(\tilde{b})] = \frac{\omega^2}{8} > 0.$$

When $B = (\sqrt{2} - 1)\omega$, in addition, $\tilde{b} = (\sqrt{2} - 1)\omega = \hat{b}$. Therefore, $E[R_f^B(\tilde{b})] = E[R_f^B(\hat{b})]$.

On the other hand,

$$E[R_{II}^B(\tilde{b})] \geq \frac{1}{\omega} (-B^2 + \omega B).$$

Differentiating $-B^2 + \omega B$ with respect to $B$, we have

$$-2B + \omega < 0.$$
Therefore, \( \frac{1}{\omega}(-B^2 + \omega B) \) is minimized at \( B = \omega \). In this case, we have

\[
E[R^B_I(\hat{b})] \geq \frac{1}{\omega}(-B^2 + \omega B) = 0.
\]

When \( B = \frac{\omega}{2} \), in addition, \( \hat{b} = 0 = \hat{b} \). Therefore, \( E[R^B_I(\hat{b})] = \frac{7\omega}{24} = E[R^B_I(\hat{b})] \).

From the above argument, a seller always obtains a positive equilibrium revenue by introducing an effective buy price.

**Proof of Proposition 4**

Now, we consider that the seller sets a reserve price \( r \in [0, \omega] \). Then, the following strategy profile is an equilibrium.

\[
\sigma^L_i(t_i) = \begin{cases} 
0 & \text{if } 0 \leq t_i \leq r, \\
t_i & \text{if } r < t_i \leq \bar{t}, \\
\bar{t} & \text{if } \bar{t} < t_i \leq \omega, \\
t_i & \text{if } \omega < t_i \leq \omega.
\end{cases}
\]

and

\[
\sigma^L_j(t_j) = \begin{cases} 
0 & \text{if } 0 \leq t_j \leq r, \\
t_j & \text{if } r < t_j \leq \bar{t}, \\
\bar{t} & \text{if } \bar{t} < t_j \leq \omega, \\
t_j & \text{if } \omega < t_j \leq \omega.
\end{cases}
\]

Note that \( r \leq \bar{t} \leq \bar{t} \leq \omega \).

We show that the strategy profile \( \sigma^L \) is a Bayesian Nash equilibrium. Consider the incentive constraints of bidder \( i \). For all \( t_i \in [0, r] \), he reduces his expected payoff by submitting a bid that is greater than the reserve price \( r \). For all \( t_i \in (r, \bar{t}] \), he reduces his expected payoff by submitting a bid that is greater or less than his type \( t_i \). For all \( t_i \in (\bar{t}, \bar{t}] \), he reduces his expected payoff by submitting another bid, for example, the bid \( \bar{t} \). For all \( t_i \in [\bar{t}, \omega] \), he reduces his expected payoff by submitting a bid that is greater or less than his type \( t_i \).

Similarly, consider the incentive constraints of bidder \( j \). For all \( t_j \in [0, r] \), he reduces his expected payoff by submitting a bid that is greater than the reserve price \( r \). For all \( t_j \in [r, \bar{t}] \), he reduces his expected payoff by submitting a bid that is greater or less than his type \( t_j \). For all \( t_j \in (\bar{t}, \bar{t}] \), he reduces his expected payoff by submitting another bid, for example, the bid \( \bar{t} \). For all \( t_j \in [\bar{t}, \omega] \), he reduces his expected payoff by submitting a bid that is greater or less than his type \( t_j \).
At the equilibrium $\sigma^L$, the seller receives the expected revenue

$$
\int_0^r \left( \int_r^\infty r f(t_j) dt_j \right) f(t_i) dt_i + \int_r^L \left( \int_t^\infty t_i f(t_j) dt_j \right) f(t_i) dt_i + \int_L^T \left( \int_t^\infty t_i f(t_j) dt_j \right) f(t_i) dt_i + \\
\int_T^1 \left( \int_t^\infty t_i f(t_j) dt_j \right) f(t_i) dt_i + \int_T^1 \left( \int_t^\infty t_j f(t_i) dt_i \right) f(t_j) dt_j + \\
\int_1^T \left( \int_t^\infty t_j f(t_i) dt_i \right) f(t_j) dt_j + \int_1^T \left( \int_t^\infty t_j f(t_i) dt_i \right) f(t_j) dt_j
$$

$$
= \frac{1}{3\omega^2} \{ \omega^3 + 3r^2\omega - 4r^3 - (\bar{t} - t)^3 \}.
$$

Here let us denote by $E[R]$ the seller’s expected revenue. Clearly, $E[R]$ is decreasing with respect to $t$, and increasing with respect to $t$. Therefore, $E[R]$ is minimized at $(\bar{t}, \bar{t}) = (r, \omega)$, and maximized at $(\bar{t}, \bar{t}) = (t, t)$ for all $t \in [r, \omega]$. Therefore, it is enough to consider, for example the strategy profile $\sigma^\gamma$ such that

$$
\sigma^\gamma_i(t_i) = \begin{cases} 
0 & \text{if } 0 \leq t_i \leq r, \\
r & \text{if } r < t_i \leq b, \\
t_i & \text{if } b < t_i \leq \omega
\end{cases}
$$

and

$$
\sigma^\gamma_j(t_j) = \begin{cases} 
0 & \text{if } 0 \leq t_j \leq r, \\
b & \text{if } r < t_j \leq b, \\
t_j & \text{if } b < t_j \leq \omega.
\end{cases}
$$

Note that $b \in [r, \omega]$.

**The maximum of the set of the equilibrium revenues**

We focus on the maximum of the set of the equilibrium revenues of each second-price, sealed-bid auction with a buy price $B$. By Propositions 2 and 3, the maximum of the set of the equilibrium revenues is

$$
E[R^B_I(0)] \quad \text{if } 0 \leq B < \frac{\omega}{2}, \\
E[R^B_I(0)] \quad \text{if } \frac{\omega}{2} \leq B \leq \omega.
$$

When $B = \frac{\omega}{2}$, in addition, $E[R^B_I(0)] = \frac{7\omega}{24} = E[R^B_{II}(0)]$.

We can regard it as a function of the buy price $B$. Let us denote by $E[R^B(B)]$ the maximum of the set of the equilibrium revenues in a second-price, sealed-bid auction with the buy price $B$. As well as the case of the minimum of the set of the equilibrium revenues, we can provide an example.

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**Example 3.** Suppose that bidders’ types are uniformly distributed on the interval [0, 1]. Then, the maximum of the set of the equilibrium revenues is

\[
E[R^w(B)] = \begin{cases} 
E[R^B_I(0)] & \text{if } 0 < B < \frac{1}{2}, \\
E[R^B_{II}(0)] & \text{if } \frac{1}{2} \leq B \leq 1, \\
\frac{1}{3} & \text{if } 1 < B.
\end{cases}
\]

Figure 3: Comparison between the maximum of the set of the equilibrium revenues \(E[R^w(B)]\) and the maximum of the set of the equilibrium revenues \(E[R^*]\)

In Figure 3, we compare the maximum of the set of the equilibrium revenues in second-price, sealed-bid auctions with a buy price \(B\) to the maximum of the set of the equilibrium revenues in second-price, sealed-bid auctions without a buy price \(B\). Figure 3 suggests that a seller can obtain the maximum equilibrium revenue by setting a buy price \(B\) which is equal to or greater than the highest possible valuation. Moreover, it is shown that a seller can obtain the same maximum equilibrium revenue as the case of second-price, sealed-bid auctions without a buy price \(B\) by properly setting a buy price \(B\).

**References**


